Combinatorial Intersection Cohomology for Fans

Gottfried Barthel, Jean-Paul Brasselet,
Karl-Heinz Fieseler, Ludger Kaup

Abstract: Intersection cohomology $IH^\bullet(X_\Delta;\mathbf{R})$ of a *complete* toric variety X_Δ, associated to a fan Δ in $\mathbf{R}^n$ and with the action of an algebraic torus $\mathbf{T} \cong (\mathbf{C}^*)^n$, is best computed using equivariant intersection cohomology $IH^\bullet_{\mathbf{T}}(X_\Delta)$. The reason is that X_Δ is IH-"equivariantly formal" and equivariant intersection cohomology provides a sheaf on X_Δ, equipped with its $\mathbf{T}$-invariant topology. An axiomatic description of that sheaf leads to the notion of a "minimal extension sheaf" $\mathcal{E}^\bullet$ on the fan Δ and a surprisingly simple, completely combinatorial approach which immediately applies to non-rational fans Δ. These sheaves are the model for a larger class of "pure" sheaves, for which we prove a "Decomposition Theorem". For a certain class of fans (including fans with convex or co-convex support), called "quasi-convex", one can define a meaningful "virtual" intersection cohomology $IH^\bullet(\Delta)$. We characterize quasi-convex fans by a purely topological condition on the support of their boundary fan $\partial\Delta$, and then deal with the question whether virtual intersection Betti numbers agree with the components of Stanley's generalized h-vector even for non-rational fans Δ, i.e. we try to prove that they satisfy the same computation algorithm. For quasi-convex fans, we prove a generalization of Stanley's formula realizing the intersection Poincaré polynomial of a complete toric variety in terms of local data. In order to show that the local data may be obtained from the virtual intersection cohomology of complete fans in lower dimensions, we have to assume that the virtual intersection cohomology of a cone σ satisfies a certain vanishing condition, analoguous to the vanishing axiom for local intersection cohomology on the closed orbit of the affine toric variety X_σ for a rational cone σ. That assumption applied to cones in dimension $n+1$ together with Poincaré duality which we show to hold for virtual intersection cohomology leads to a Hard Lefschetz theorem for polytopal fans and to the desired second step in the computation algorithm for virtual intersection Poincaré polynomials.

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Introduction

A basic combinatorial invariant of a *complete simplicial fan* Δ in $\mathbf{R}^n$ is its *h-vector*, encoding the numbers of cones of given dimension. By the classical *Dehn-Sommerville relations*, the equality $h_i = h_{n-i}$ holds, i.e., the vector is *palindromic*.

If Δ is *rational*, then the h-vector admits a topological interpretation in terms of the associated compact $\mathbf{Q}$-smooth toric variety X_Δ: By the theorem of Jurkiewicz and Danilov, the real [1)] cohomology algebra $H(X_\Delta)$ is a quotient of the Stanley-Reisner ring of Δ. In particular, $H(X_\Delta)$ is a combinatorial invariant of Δ, it "lives" only in even degrees, and $h_i(\Delta)$ equals the Betti number $b_{2i}(X_\Delta)$. Since simplicial fans are combinatorially equivalent to rational ones, this interpretation allows to apply topological results to combinatorics. Thus, the Dehn-Sommerville equations are just a combinatorial version of Poincaré duality (PD). As a deeper application, we mention Stanley's proof of the necessity of McMullen's conditions that characterize the possible h-vectors of simplicial *polytopal* fans: It involves the "Hard" Lefschetz theorem that holds since the corresponding toric variety is projective.

In the *non-rational* case, we may "reverse" the theorem of Jurkiewicz and Danilov and take the quotient of the Stanley-Reisner ring as definition of a "virtual cohomology algebra" of the fan Δ, thus obtaining virtual Betti numbers $b_{2i}(\Delta)$ that coincide with $h_i(\Delta)$ for $0 \le i \le n$.

In the *rational non-simplicial* case, using the Betti numbers of the associated toric variety as a definition of the h-vector no longer gives an invariant with the analoguous properties, in fact Poincaré duality fails to hold, and even worse, it is not determined by the structure of Δ as a partially ordered set only. Instead, in order to get an invariant which shares the nice properties the classical h-vector has in the simplicial case, one has to replace singular homology with intersection cohomology: The i-th component of the *generalized h-vector* is defined as $h_i(\Delta) := \dim IH^{2i}(X_\Delta)$, i.e., equals the $2i$-th intersection Betti number of X_Δ. It satisfies Poincaré duality and its components are linear functions in the numbers of flags of cones with prescribed sequences of dimensions. Its computation can be done recursively using a two step induction algorithm involving the g-vector $(g_0(\sigma), \ldots, g_r(\sigma))$ of a cone σ, where $g_i(\sigma) := \dim IH^{2i}(X_\sigma/\mathbf{T}'_\sigma)$. Here $\mathbf{T}'_\sigma \subset \mathbf{T}$ denotes a complementary subtorus to the stabilizer of a point in the closed orbit of X_σ and $r := [\dim \sigma/2] - 1$. In fact, that algorithm is used to define the generalized h- and the g-vector also for non-rational cones and fans, cf. [S].

In our article [Hi], we have proved that in this situation, the rôle of the Stanley-Reisner ring is played by the $A^\bullet := S^\bullet(V^*)$-module $\mathcal{E}^\bullet(\Delta)$ of global sections of a so-called "minimal extension sheaf" $\mathcal{E}^\bullet$ on the "fan space" Δ. (In the simplicial case

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[1)] For ease of exposition, we use real coefficients.

$\mathcal{E}^\bullet(\Delta)$ coincides with the $A^\bullet$-algebra of piecewise-polynomial functions on $|\Delta|$, which for complete Δ is nothing but the Stanley-Reisner ring.) That motivates in the non-rational case the following definition of the *virtual intersection cohomology* $IH^\bullet(\Delta)$ of a fan Δ: One sets $IH^\bullet(\Delta) := A^\bullet/\mathbf{m} \otimes_{A^\bullet} \mathcal{E}^\bullet(\Delta)$, where $\mathbf{m} := A^{>0}$, and hopes that for complete Δ the components of the generalizd h-vector turn out to be the virtual intersection Betti numbers of Δ, i.e., $h_i(\Delta) = \dim_{\mathbf{R}} IH^{2i}(\Delta)$.

In this article we start the investigation of the algebraic theory of such minimal extension sheaves which hopefully in the near future will lead to the proof of the above interpretation of the components of the generalized h-vector. In the first part we give the definition of minimal extension sheaves and recall the results of [Hi], where the virtual intersection Betti numbers of a complete rational fan Δ are seen to equal the intersection Betti numbers of X_Δ. The second section is devoted to combinatorially pure sheaves over the "fan space" Δ: They turn out to be direct sums of simple sheaves, which are generalized minimal extension sheaves: To each cone $\tau \in \Delta$ we associate a simple pure sheaf ${}_\tau\mathcal{E}^\bullet$, such that $\mathcal{E}^\bullet = {}_o\mathcal{E}^\bullet$ with the zero cone o, and prove a decomposition theorem (Theorem 2.3) for pure sheaves. As a corollary, we obtain a proof of Kalai*s conjecture for virtual intersection cohomology Poincaré polynomials, as proposed by Tom Braden, cf. also [BrMPh].

In the third section, we characterize "quasi-convex" fans, i.e., those fans Δ for which the $A^\bullet$-module $\mathcal{E}^\bullet(\Delta)$ is free. In fact, a purely n-dimensional fan Δ is quasi-convex if and only if the support of its boundary fan $|\partial\Delta|$ is a real homology manifold, see Theorem 3.8/9, so in particular fans with convex or co-convex support (i.e. $|\Delta|$ resp. $V \setminus |\Delta|$ are convex) are quasi-convex. For rational fans Δ, quasi-convexity is a necessary and sufficient condition in order that the virtual intersection cohomology agrees with the ordinary intersection cohomology of the associated toric variety X_Δ, i.e. $IH^\bullet(\Delta) \cong IH^\bullet(X_\Delta)$. Another equivalent reformulation of that fact is that the intersection Betti numbers of the associated toric variety X_Δ vanish in odd degrees. On the other hand the freeness condition is essential in order to have a satisfactory duality theory both on $\mathcal{E}^\bullet(\Delta)$ and $IH^\bullet(\Delta) = A^\bullet/\mathbf{m} \otimes_{A^\bullet} \mathcal{E}^\bullet(\Delta)$. In fact, quasi-convexity turns out to be equivalent to the acyclicity of the cellular cochain complex with coefficients in the sheaf $\mathcal{E}^\bullet$, see Theorem 3.8.

The fourth section deals with the computation of the virtual intersection Poincaré polynomials $P_\Delta := \sum \dim IH^{2i}(\Delta) t^{2i}$: For a quasi-convex fan Δ the polynomial P_Δ can be expressed in terms of the local Poincaré polynomials P_σ, see Theorem 4.3, where P_σ denotes the virtual intersection Poincaré polynomial of the fan consisting of σ and its proper faces. That is a consequence of the above mentioned acyclicity of the cellular complex, and the fact that $IH^\bullet(\Delta)$ and $\mathcal{E}^\bullet(\Delta)$ as well as $IH^\bullet(\sigma)$ and $\mathcal{E}^\bullet(\sigma)$ are related by Künneth type formulae. In order to get a computation algorithm for P_Δ we have to relate P_σ to P_Λ, the Poincaré polynomial of the "flattened

boundary fan" $\Lambda = \Lambda_\sigma$ of σ, which in fact is a polytopal fan in $V/V_\sigma, V_\sigma := \mathrm{span}(\sigma)$. For that step we need the vanishing condition $IH^q(\sigma) = 0$ for $q \geq \dim \sigma > 0$. In the case of a rational cone that condition turns out to be equivalent to the vanishing condition for the local intersection cohomology of X_σ along its closed orbit; in fact we conjecture it to hold in general, but up to now have to state it as a condition $V(\sigma)$, see 1.7, in the non-rational case. The above vanishing condition together with Poincaré duality (see section 5) leads to a "Hard Lefschetz Theorem" for the virtual intersection cohomology $IH^\bullet(\Lambda)$ of the polytopal fan Λ_σ, see Theorem 4.6, and that theorem is behind the description of P_σ in terms of P_Λ. In particular, if all the cones in Δ satisfy the above vanishing condition, we have $h_i(\Delta) = \dim IH^{2i}(\Delta)$.

Finally the last section is devoted to Poincaré duality: On a minimal extension sheaf $\mathcal{E}^\bullet$ we define a - non-canonical - internal intersection product $\mathcal{E}^\bullet \times \mathcal{E}^\bullet \to \mathcal{E}^\bullet$, that composed with an evaluation map leads to duality isomorphisms $\mathcal{E}^\bullet(\Delta) \cong \mathcal{E}^\bullet(\Delta, \partial\Delta)^*$ as well as $IH^\bullet(\Delta) \cong IH^\bullet(\Delta, \partial\Delta)^*$, see Theorem 5.3.

In order to make our results accessible to non-specialists, we have aimed at avoiding technical "machinery" and keeping the presentation as elementary as possible. Many essential results of the present article are contained in Chapters 7–10 of our Uppsala preprint [2]; the current version has been announced in the note [Fi$_2$]. Using the formalism of derived categories, closely related work has been done by Tom Braden in the rational case and by Paul Bressler and Valery Lunts in the polytopal case. Tom Braden sent us a manuscript presented at the AMS meeting in Washington, January 2000. Even more recently, Paul Bressler and Valery Lunts published their ideas in the e-print [BreLu$_2$].

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0. Preliminaries

0.A Cones and Fans: Let V be a real vector space of dimension n. A non-zero linear form $\alpha: V \to \mathbf{R}$ on V determines the *upper halfspace* $H_\alpha := \{v \in V; \alpha(v) \geq 0\}$. A (strictly convex polyhedral) *cone* in V is a finite intersection $\sigma = \bigcap_{i=1}^r H_{\alpha_i}$ of halfspaces with linear forms satisfying $\bigcap_{i=1}^r \ker \alpha_i = \{0\}$. We let $V_\sigma := \sigma + (-\sigma)$ denote the linear span of σ in V, and define $\dim \sigma := \dim V_\sigma$. A cone of dimension d is often called a *d-cone* for short.

A cone also may be described as the set $\sigma = \sum_{j=1}^s \mathbf{R}_{\geq 0} v_j$ of all positive linear combinations of a finite set of non-zero vectors v_j in V. In particular, a cone spanned by a linearly independent system of generators is called *simplicial*. Cones of dimen-

[2] *"Equivariant Intersection Cohomology of Toric Varieties"*, UUDM report 1998:34

sion $d \le 2$ are always simplicial; in particular, this applies to the *zero cone* $o := \{0\}$ and to every *ray* (i.e., a one-dimensional cone $\mathbf{R}_{\ge 0} v$).

A *face* of a cone σ is any intersection $\tau = \sigma \cap \ker \beta$, where $\beta \in V^*$ is a linear form with $\sigma \subset H_\beta$. We then write $\tau \preceq \sigma$ (and $\tau \prec \sigma$ for a *proper* face). If in addition $\dim \tau = \dim \sigma - 1$, we call τ a *facet* of σ and write $\tau \prec_1 \sigma$.

A *fan* in V is a non-empty finite set Δ of cones such that each face τ of a cone $\sigma \in \Delta$ also belongs to Δ and the intersection $\sigma \cap \sigma'$ of two cones $\sigma, \sigma' \in \Delta$ is a face of both, σ and σ'. We say that Δ is generated by the cones $\sigma_1, \ldots, \sigma_r$, if Δ consists of all the cones which are a face of some cone $\sigma_i, 1 \le i \le r$. In particular, a given cone σ *generates* the fan $\langle \sigma \rangle$ consisting of σ and its proper faces; such a fan is also called an *affine fan* and occasionally is simply denoted σ. Moreover, we associate to σ its *boundary fan* $\partial\sigma := \langle \sigma \rangle \setminus \{\sigma\}$.

Every fan is generated by the collection $\Delta^{\max}$ consisting of its maximal cones. We define

$$\Delta^k := \{\sigma \in \Delta \; ; \; \dim \sigma = k\} \qquad \text{and} \qquad \Delta^{\le k} := \bigcup_{r \le k} \Delta^r ,$$

the latter being a subfan called the *k-dimensional skeleton* (or *k-skeleton* for short). The fan Δ is called *purely n-dimensional* if $\Delta^{\max} = \Delta^n$. In that case, its *boundary fan* $\partial\Delta$ is generated by those $(n-1)$-cones that are facets of precisely one n-cone in Δ. A fan is called *simplicial* if all its cones are simplicial; this holds if and only if it is generated by simplicial cones.

The support $|\Delta| := \cup_{\sigma \in \Delta} \, \sigma \subset V$ is the union of all the cones in Δ, and Δ is called *complete* if and only if $|\Delta| = V$. We remark that the boundary fan $\partial\Delta$ of a purely n-dimensional fan Δ is supported by the topological boundary of $|\Delta|$. – A fan Δ in V is called *N-rational* if there exists a lattice (i.e., a discrete additive subgroup) $N \subset V$ of maximal rank such that $\varrho \cap N \ne \{0\}$ for each ray $\varrho \in \Delta$.

A *subfan* Λ of a fan Δ is any subset that itself is a fan; we then write $\Lambda \preceq \Delta$ (and $\Lambda \prec \Delta$ if in addition Λ is a *proper* subfan). The collection of all subfans of Δ clearly satisfies the axioms for the open sets of a topology on Δ (The empty set is admitted as a subfan). In the sequel, we always endow Δ with this *fan topology* and consider it as a topological space, the *fan space*.

0.B Graded $A^\bullet$-modules: In this subsection, we recall algebra results useful for the sequel. We denote with $A^\bullet$ the symmetric algebra $S^\bullet(V^*)$ over the dual vector space V^* of V. Its elements are canonically identified with polynomial functions on V. In the case of a rational fan $A^\bullet$ is isomorphic to the cohomology algebra $H^\bullet(B\mathbf{T})$ of the classifying space $B\mathbf{T}$ of the complex algebraic n-torus $\mathbf{T} \cong (\mathbf{C}^*)^n$ acting on the associated toric variety. Motivated by that topological considerations, we endow $A^\bullet$ with the positive even grading determined by setting $A^{2q} := S^q(V^*)$; in particular,

$A^2 = V^*$ consists of all linear forms on V. For a cone σ in V, let $A^\bullet_\sigma$ denote the algebra $S^\bullet(V^*_\sigma)$ with the grading as above. The natural projection $V^* \to V^*_\sigma$ extends to an epimorphism $A^\bullet \to A^\bullet_\sigma$ of graded algebras. We usually consider the elements in $A^\bullet_\sigma$ as functions $f\colon \sigma \to \mathbf{R}$; the above epimorphism then corresponds to the restriction of polynomial functions. If σ is of dimension n, then the equality $A^\bullet_\sigma = A^\bullet$ clearly holds.

For a graded $A^\bullet$-module $F^\bullet$, we let $\overline{F}^\bullet$ denote its residue class module

(0.B.1) $$\overline{F}^\bullet \;:=\; F^\bullet/(\mathbf{m}\cdot F^\bullet) \;\cong\; \mathbf{R}^\bullet \otimes_{A^\bullet} F^\bullet \,,$$

where $\mathbf{m} := A^{>0} \subset A^\bullet$ is the unique homogeneous maximal ideal of $A^\bullet$ and where $\mathbf{R}^\bullet := A^\bullet/\mathbf{m} = \overline{A}^\bullet$ is the field $\mathbf{R}$, considered as graded algebra concentrated in degree zero. Clearly $\overline{F}^\bullet$ is a graded vector space over $\mathbf{R}$ that is finite dimensional if $F^\bullet$ is finitely generated over $A^\bullet$. If $F^\bullet$ is bounded from below , then the reverse implication holds, more precisely, a family $(f_1, \ldots, f_r)$ of homogeneous elements in $F^\bullet$ generates $F^\bullet$ over $A^\bullet$ if and only if the system of residue classes $(\overline{f}_1, \ldots, \overline{f}_r)$ modulo $\mathbf{m}\cdot F^\bullet$ generates the vector space $\overline{F}^\bullet$. In that case, we have $\mathrm{rk}_{A^\bullet} F^\bullet \le \dim \overline{F}^\bullet$, with equality holding if and only if $F^\bullet$ is a free $A^\bullet$-module. The collection $(f_1, \ldots, f_r)$ is part of a basis of the free $A^\bullet$-module $F^\bullet$ over $A^\bullet$ if and only if $(\overline{f}_1, \ldots, \overline{f}_r)$ is linearly independent over $\mathbf{R}$. Furthermore, every homomorphism $\varphi\colon F^\bullet \to G^\bullet$ of graded $A^\bullet$-modules induces a homomorphism $\overline{\varphi}\colon \overline{F}^\bullet \to \overline{G}^\bullet$ of graded vector spaces that is surjective if and only if φ is so. If $F^\bullet$ is free, then every homomorphism $\psi\colon \overline{F}^\bullet \to \overline{G}^\bullet$ can be lifted to a homomorphism $\varphi\colon F^\bullet \to G^\bullet$ (i.e., $\overline{\varphi} = \psi$ holds); if $G^\bullet$ is free, then φ is an isomorphism if and only if that holds for $\overline{\varphi}$.

A finitely generated $A^\bullet$-module $F^\bullet$ is *free* if and only if $\mathrm{Tor}_1^{A^\bullet}(F^\bullet, \mathbf{R}^\bullet) = 0$. That condition is obviously necessary, so let us show that it is also sufficient: As we have seen above, there is a surjection $(A^\bullet)^m \to F^\bullet$ where $m := \dim \overline{F}^\bullet$; let $K^\bullet$ be its kernel. Since $\mathrm{Tor}_1^{A^\bullet}(F^\bullet, \mathbf{R}^\bullet) = 0$, the exact sequence

$$0 \longrightarrow K^\bullet \longrightarrow (A^\bullet)^m \longrightarrow F^\bullet \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \overline{K}^\bullet \longrightarrow (\overline{A}^\bullet)^m \longrightarrow \overline{F}^\bullet \longrightarrow 0\,.$$

By construction, $(\overline{A}^\bullet)^m \to \overline{F}^\bullet$ is an isomorphism, so we have $\overline{K}^\bullet = 0$ and thus also $K^\bullet = 0$, i.e., $F^\bullet \cong (A^\bullet)^m$ is free. □

By means of the restriction map $A^\bullet \to A^\bullet_\sigma$, an $A^\bullet_\sigma$-module $F^\bullet_\sigma$ is an $A^\bullet$-module, and there is a natural isomorphism $\overline{F}^\bullet_\sigma \cong F^\bullet_\sigma/(\mathbf{m}_\sigma \cdot F^\bullet_\sigma)$. Let us denote by $V_\sigma^\perp$ the orthogonal complement of $V_\sigma \subset V$ in the dual vector space V^*, we remark that, using the Koszul complex for the $A^\bullet$-module $I(V_\sigma) := A^\bullet \cdot V_\sigma^\perp \subset A^\bullet$, one finds a natural isomorphism of vector spaces

(0.B.2) $$\mathrm{Tor}_i^{A^\bullet}(A^\bullet_\sigma, \mathbf{R}^\bullet) \cong \Lambda^i V_\sigma^\perp$$

over $\mathbf{R}^\bullet = A^\bullet/\mathbf{m}$.

0.C Sheaves on a fan space: Let $\mathcal{F}$ be a sheaf of real vector spaces on the fan space Δ. Since the "affine" open sets $\langle\sigma\rangle \preceq \Delta$ form a basis of the topology of Δ, the sheaf $\mathcal{F}$ is uniquely determined by the collection of its values $\mathcal{F}(\sigma) := \mathcal{F}\big(\langle\sigma\rangle\big)$ for $\sigma \in \Delta$, together with the restriction homomorphisms $\varrho^\sigma_\tau \colon \mathcal{F}(\sigma) \to \mathcal{F}(\tau)$ for $\tau \preceq \sigma$. On the other hand, every such collection belongs to a sheaf $\mathcal{F}$ on Δ, since an "affine" open subset $\langle\sigma\rangle$ can not be covered by strictly smaller open sets. Furthermore, we note that the sheaf $\mathcal{F}$ is flabby if and only if every restriction map $\mathcal{F}(\sigma) \to \mathcal{F}(\partial\sigma)$ is surjective. – In the same spirit of ideas, sheaves on a fan occur in the work of Bressler and Lunts [BreLu$_2$], Brion [Bri] and McConnell [McC].

In particular, we consider the sheaf $\mathcal{A}^\bullet$ of graded algebras on Δ given by $\mathcal{A}^\bullet(\sigma) := A^\bullet_\sigma$, the restriction homomorphisms $\varrho^\sigma_\tau \colon A^\bullet_\sigma \to A^\bullet_\tau$ being the natural maps $S^\bullet(V^*_\sigma) \to S^\bullet(V^*_\tau)$ induced by the inclusions $V_\tau \hookrightarrow V_\sigma$ of a face $\tau \preceq \sigma$. The sections $\mathcal{A}^\bullet(\Lambda)$ on a subfan $\Lambda \preceq \Delta$ constitute the algebra of (Λ-) piecewise polynomial functions on Λ in a natural way.

If $\mathcal{F}^\bullet$ is a sheaf of $\mathcal{A}^\bullet$-modules, then every $\mathcal{F}^\bullet(\Lambda)$ also is an $A^\bullet$-module, and if $\mathcal{F}^\bullet(\sigma)$ is finitely generated for every cone $\sigma \in \Delta$, then so is $\mathcal{F}(\Lambda)$ for every subfan $\Lambda \preceq \Delta$: That is an immediate consequence of the fact that $A^\bullet$ is a noetherian ring and of the inclusion $\mathcal{F}^\bullet(\Lambda) \subset \bigoplus_{\sigma\in\Lambda^{\max}} \mathcal{F}^\bullet(\sigma)$.

For notational convenience, we often write

$$F^\bullet_\Lambda := \mathcal{F}^\bullet(\Lambda) \qquad \text{and} \qquad F^\bullet_\sigma := \mathcal{F}^\bullet(\langle\sigma\rangle)\ ;$$

more generally, for a pair of subfans (Λ, Λ_0), we define

$$F^\bullet_{(\Lambda,\Lambda_0)} := \ker(\varrho^\Lambda_{\Lambda_0} \colon F^\bullet_\Lambda \longrightarrow F^\bullet_{\Lambda_0})$$

to be the submodule of sections on Λ vanishing on Λ_0. In particular, for a purely n-dimensional fan Δ, we obtain in that way the module

$$F^\bullet_{(\Delta,\partial\Delta)} := \ker(\varrho^\Delta_{\partial\Delta} \colon F^\bullet_\Delta \longrightarrow F^\bullet_{\partial\Delta})$$

of sections over Δ with "compact supports".

Sheaves and residue class sheaves: To a sheaf $\mathcal{F}^\bullet$ of $\mathcal{A}^\bullet$-modules, we associate the sheaf $\overline{\mathcal{F}}^\bullet$ determined by the assignment $\sigma \mapsto \overline{F}^\bullet_\sigma = \overline{\mathcal{F}^\bullet(\sigma)}$. This is a sheaf of graded $\mathbf{R}^\bullet$-modules, i.e., of graded real vector spaces. The sheaf $\overline{\mathcal{F}}^\bullet$ is associated to the *presheaf* determined by the assignment $\Lambda \mapsto \overline{\mathcal{F}^\bullet(\Lambda)}$, which in general is not a sheaf: For a non-affine fan Δ, the canonical homomorphism $\overline{\mathcal{F}^\bullet(\Delta)} \to \overline{\mathcal{F}}^\bullet(\Delta)$ need not be an isomorphism. As an example, let Δ be the fan describing the projective line $\mathbf{P}_1$. Then the real vector space $\overline{\mathcal{A}}^\bullet(\Delta) = \mathbf{R}^\bullet$ is one-dimensional and concentrated in degree 0, while $\overline{A}^\bullet_\Delta = \overline{\mathcal{A}^\bullet(\Delta)} \cong H^\bullet(\mathbf{P}_1)$ is the direct sum of two one-dimensional weight subspaces in degree 0 and 2.

0.D Fan constructions associated with a cone: In addition to the *affine fan* $\langle\sigma\rangle$ and the *boundary fan* $\partial\sigma$ associated with a cone σ, we need two more constructions. Firstly, if σ belongs to a fan Δ, we consider the *star*

$$\mathrm{st}_\Delta(\sigma) := \{\gamma \in \Delta \,;\, \sigma \preceq \gamma\} \tag{0.D.0}$$

of σ in Δ. This set is not a subfan of Δ – we note in passing that it is the *closure* of the one-point set $\{\sigma\}$ in the fan topology –, but its image

$$\Delta_\sigma \;:=\; p\big(\mathrm{st}_\Delta(\sigma)\big) \;=\; \{p(\gamma) \,;\, \sigma \preceq \gamma\} \tag{0.D.1}$$

under the quotient projection $p\colon V \to V/V_\sigma$ is a fan in V/V_σ, called the *"transversal fan"* of σ in Δ.

Secondly, if σ is a non-zero cone σ, we consider its *"flattened boundary fan"*, the fan $\Lambda_\sigma = \Lambda_\sigma(L)$ that is obtained by projecting the boundary fan $\partial\sigma$ onto the quotient vector space V_σ/L, where L is a line in V passing through the relative interior $\mathring{\sigma}$: If $\pi\colon V_\sigma \to V_\sigma/L$ is the quotient projection, then we pose

$$\Lambda_\sigma := \pi(\partial\sigma) = \{\pi(\tau) \,;\, \tau \prec \sigma\} \ . \tag{0.D.2}$$

This fan is *complete*. Restricting the projection π to the support of $\partial\sigma$ yields a (piecewise linear) homeomorphism

$$\pi|_{\partial\sigma}\colon |\partial\sigma| \longrightarrow |\Lambda_\sigma| = V_\sigma/L$$

that in turn induces a homeomorphism $\partial\sigma \to \Lambda_\sigma$ of fan spaces; in particular, the combinatorial type of Λ_σ is independent of the choice of L. Any linear function $T \in A^2_\sigma$ not identically vanishing on L provides an isomorphism $L \xrightarrow{\cong} \mathbf{R}$; furthermore, it gives rise to a decomposition $V_\sigma = \ker(T) \oplus L$ and hence, to an isomorphism $\ker(T) \cong V_\sigma/L$. Identifying V_σ and $(V_\sigma/L) \times \mathbf{R}$ via these isomorphisms, we see that the support $|\partial\sigma|$ of the boundary fan is the graph of the strictly convex Λ_σ-piecewise linear function $f := T \circ (\pi|_{\partial\sigma})^{-1}\colon V_\sigma/L \to \mathbf{R}$.

On the other hand, for a complete fan Λ in a vector space W and a Λ-strictly convex piecewise linear function $f\colon W \to \mathbf{R}$, the convex hull of the graph Γ_f in $W \times \mathbf{R}$ is a cone $\gamma := \gamma^+(f)$ with boundary $\partial\gamma = \Gamma_f$.

1. Minimal Extension Sheaves

The investigation of a "virtual" intersection cohomology theory for arbitrary fans is couched in terms of a certain class of sheaves on fans that we call *minimal extension sheaves.* In this section, we introduce that notion and study some elementary properties of such sheaves.

1.1 Definition. *A sheaf $\mathcal{E}^\bullet$ of graded $\mathcal{A}^\bullet$-modules on the fan Δ is called a minimal extension sheaf (of $\mathbf{R}^\bullet$) if it satisfies the following conditions:*

(N) **Normalization:** *One has* $E^\bullet_o \cong A^\bullet_o = \mathbf{R}^\bullet$ *for the zero cone* o.

(PF) **Pointwise Freeness:** *For each cone* $\sigma \in \Delta$*, the module* $E^\bullet_\sigma$ *is free over* $A^\bullet_\sigma$.

(LME) **Local Minimal Extension** mod $\mathbf{m}$**:** *For each cone* $\sigma \in \Delta \setminus \{0\}$*, the restriction mapping*

$$\varrho_\sigma := \varrho^\sigma_{\partial\sigma} \colon E^\bullet_\sigma \longrightarrow E^\bullet_{\partial\sigma}$$

induces an isomorphism

$$\overline{\varrho}_\sigma \colon \overline{E}^\bullet_\sigma \xrightarrow{\cong} \overline{E}^\bullet_{\partial\sigma}$$

of graded real vector spaces.

The above condition (LME) implies that $\mathcal{E}^\bullet$ is minimal in the set of all flabby sheaves of graded $\mathcal{A}^\bullet$-modules satisfying conditions (N) and (PF), whence the name "minimal extension sheaf":

1.2 Remark. *Let* $\mathcal{E}^\bullet$ *be a minimal extension sheaf on a fan* Δ.

i) The sheaf $\mathcal{E}^\bullet$ *is flabby and vanishes in odd degrees.*

ii) For each subfan $\Lambda \preceq \Delta$*, the* $A^\bullet$*-module* $E^\bullet_\Lambda$ *is finitely generated. For each cone* $\sigma \in \Delta$*, there is an isomorphim of graded* $A^\bullet_\sigma$*-modules*

$$E^\bullet_\sigma \;\cong\; A^\bullet_\sigma \otimes_{\mathbf{R}} \overline{E}^\bullet_\sigma\ . \tag{1.2.1}$$

Proof: (i) By the results of 0.B, condition (LME) implies that ϱ_σ is surjective for every cone $\sigma \in \Delta$; hence, 0.C asserts flabbiness.

(ii) Let us assume that $E^\bullet_\tau$ is finitely generated for $\dim\tau \le k$, then so is $E^\bullet_\Lambda$ for every subfan $\Lambda \preceq \Delta^{\le k}$, see 0.C. In particular, if σ is a cone of dimension $k+1$, then $E^\bullet_{\partial\sigma}$ is finitely generated, whence $\overline{E}^\bullet_\sigma \cong \overline{E}^\bullet_{\partial\sigma}$ is finite-dimensional, and thus the free $A^\bullet_\sigma$-module $E^\bullet_\sigma$ is finitely generated. Now apply 0.C. Since $A^\bullet$ only lives in even degrees, the obvious $\mathbf{R}^\bullet$-splitting $F^\bullet = F^{\mathrm{even}} \oplus F^{\mathrm{odd}}$ of a graded $A^\bullet$-module actually is a decomposition into graded $A^\bullet$-submodules. Hence, a finitely generated $A^\bullet$-module $F^\bullet$ vanishes in odd degrees if and only if $\overline{F}^\bullet$ does. Thus, we may use induction on the k-skeletons of Δ as above. — The isomorphism (1.2.1) now is an immediate consequence of the results quoted in (0.B) since the $A^\bullet_\sigma$-module $E^\bullet_\sigma$ is free and finitely generated.

We now prove that on an arbitrary fan Δ, a minimal extension sheaf can be constructed recursively and that it is unique up to isomorphism; hence, we may speak of *the* minimal extension sheaf $\mathcal{E}^\bullet := {}_\Delta\mathcal{E}^\bullet$ of Δ.

1.3 Proposition (Existence and Uniqueness of Minimal Extension Sheaves): *On an arbitrary fan* Δ*, there exists a minimal extension sheaf* $\mathcal{E}^\bullet$*; it is unique up to an isomorphism of graded* $\mathcal{A}^\bullet$*-modules. More precisely, for any two such sheaves* $\mathcal{E}^\bullet$ *and* $\mathcal{F}^\bullet$ *on* Δ*, every isomorphism* $E^\bullet_o \cong F^\bullet_o$ *extends to an isomorphism* $\mathcal{E}^\bullet \xrightarrow{\cong} \mathcal{F}^\bullet$ *of graded* $\mathcal{A}^\bullet$*-modules.*

As to the uniqueness of such an extension, see Remark 1.8, (iii).

Proof: For the *existence*, we define the sheaf $\mathcal{E}^\bullet$ inductively on the k-skeleton subfans $\Delta^{\le k}$, starting with $E^\bullet_o := \mathbf{R}^\bullet$ for $k = 0$. For $k > 0$, we assume that $\mathcal{E}^\bullet$ has been defined on $\Delta^{<k}$; in particular, $E^\bullet_{\partial\sigma}$ exists for every cone $\sigma \in \Delta^k$. It thus suffices to define $E^\bullet_\sigma$, together with a restriction homomorphism $E^\bullet_\sigma \to E^\bullet_{\partial\sigma}$. According to (1.2.1), we set $E^\bullet_\sigma \ := \ A^\bullet_\sigma \otimes_{\mathbf{R}} \overline{E}^\bullet_{\partial\sigma}$, and define the restriction map using a $\mathbf{R}^\bullet$-linear section $s\colon \overline{E}^\bullet_{\partial\sigma} \to E^\bullet_{\partial\sigma}$ of the residue class map $E^\bullet_{\partial\sigma} \to \overline{E}^\bullet_{\partial\sigma}$.

For the *uniqueness* of minimal extension sheaves up to isomorphism, we use the same induction pattern and show how a given isomorphism $\varphi\colon \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ of such sheaves on $\Delta^{<k}$ may be extended to Δ^k. It suffices to verify that, for each cone $\sigma \in \Delta^k$, there is a lifting of $\varphi_{\partial\sigma}\colon E^\bullet_{\partial\sigma} \xrightarrow{\cong} F^\bullet_{\partial\sigma}$ to an isomorphism $\varphi_\sigma\colon E^\bullet_\sigma \xrightarrow{\cong} F^\bullet_\sigma$. Using the results recalled in section 0.B, the existence of such a lifting follows easily from the properties of graded $A^\bullet_\sigma$-modules: We choose a homogeneous basis $(e_1, \ldots, e_r)$ of the free $A^\bullet_\sigma$-module $E^\bullet_\sigma$. Since $\mathcal{F}^\bullet$ is a flabby sheaf, the images $\varphi_{\partial\sigma}(e_i|_{\partial\sigma})$ in $F^\bullet_{\partial\sigma}$ can be extended to homogeneous sections $f_1, \ldots, f_r$ in $F^\bullet_\sigma$ with $\deg e_j = \deg f_j$. The induced restriction isomorphism $\overline{F}^\bullet_\sigma \xrightarrow{\cong} \overline{F}^\bullet_{\partial\sigma}$ maps the residue classes $\overline{f}_1, \ldots, \overline{f}_r$ to a basis of $\overline{F}^\bullet_{\partial\sigma}$. It is now clear that these sections $f_1, \ldots, f_r$ form a basis of the free $A^\bullet_\sigma$-module $F^\bullet_\sigma$, and that $e_i \mapsto f_i$ defines a lifting $\varphi_\sigma\colon E^\bullet_\sigma \xrightarrow{\cong} F^\bullet_\sigma$ of $\varphi_{\partial\sigma}$. □

Simplicial fans are easily characterized in terms of minimal extension sheaves.

1.4 Proposition: *The following conditions for a fan Δ are equivalent:*

i) Δ *is simplicial,*

ii) $\mathcal{A}^\bullet$ *is a minimal extension sheaf on* Δ.

Proof: "(ii) $\Longrightarrow$ (i)" Assuming that $\mathcal{A}^\bullet$ is a minimal extension sheaf, we show by induction on the dimension d that for each cone $\sigma \in \Delta^d$, the number k of its rays equals d, i.e., that σ is simplicial. This is always true for $d \le 2$. As induction hypothesis, we assume that the boundary fan $\partial\sigma$ is simplicial. On each ray of σ, we choose a non-zero vector v_i. Then there exist unique piecewise linear functions $f_i \in A^2_{\partial\sigma}$ with $f_i(v_j) = \delta_{ij}$ for $i, j = 1, \ldots, k$. As these functions $f_1, \ldots, f_k$ are clearly linearly independent over $\mathbf{R}$, we have $\dim_{\mathbf{R}} A^2_{\partial\sigma} \ge k$.

Since $\overline{A}^\bullet_\sigma = \mathbf{R}^\bullet$, we have $\overline{A}^2_\sigma = 0$. Furthermore, we note that the homogeneous component of degree 2 in the graded module $\mathbf{m}A^\bullet_{\partial\sigma}$ is nothing but $A^2 \cdot A^0_{\partial\sigma} = A^2|_{\partial\sigma} = A^2_\sigma|_{\partial\sigma}$. Since $\mathcal{A}^\bullet$ is a minimal extension sheaf by assumption, the induced restriction homomorphism $\overline{A}^\bullet_\sigma \to \overline{A}^\bullet_{\partial\sigma}$ is an isomorphism. We thus obtain equalities

$$0 \ = \ \overline{A}^2_\sigma \ = \ \overline{A}^2_{\partial\sigma} \ = \ A^2_{\partial\sigma}/(A^2_\sigma|_{\partial\sigma}) \ ,$$

in particular yielding $d \le k \le \dim A^2_{\partial\sigma} = \dim A^2_\sigma|_{\partial\sigma}$. As $\partial\sigma$ spans V_σ, we further have $\dim A^2_\sigma|_{\partial\sigma} = \dim A^2_\sigma = d$, thus yielding the desired result $k = d$.

"(i) $\Longrightarrow$ (ii)": We again proceed by induction on the dimension d, proving that for any simplicial cone σ with $\dim \sigma = d$ a minimal extension sheaf $\mathcal{E}^\bullet$ on $\langle\sigma\rangle$ is naturally isomorphic to the sheaf $\mathcal{A}^\bullet$. The case $d = 0$ being immediate, let us first remark that a simplicial cone is the sum $\sigma = \tau + \varrho$ of any facet $\tau \prec_1 \sigma$ and the remaining ray ϱ that is not contained in V_τ. Using the decomposition $V_\sigma = V_\tau \oplus V_\varrho$ and the corresponding projections $p : V_\sigma \longrightarrow V_\tau$ and $q : V_\sigma \longrightarrow V_\varrho$ we can write $A^\bullet_\sigma \cong B^\bullet_\tau \otimes_{\mathbf{R}} B^\bullet_\varrho$ with the subalgebras

$$B^\bullet_\tau := p^*(S(V^*_\tau)) \qquad \text{and} \qquad B^\bullet_\varrho := q^*(S(V^*_\varrho)) \; . \tag{1.4.1}$$

Then according to Lemma 1.5, we have isomorphisms

$$E^\bullet_\sigma \;\cong\; A^\bullet_\sigma \otimes_{B^\bullet_\tau} E^\bullet_\tau \;\cong\; A^\bullet_\sigma \otimes_{B^\bullet_\tau} B^\bullet_\tau \;=\; A^\bullet_\sigma \; ,$$

as the facet τ is simplicial and hence we have $E^\bullet_\tau \cong B^\bullet_\tau$ by induction hypothesis. □

1.5 Lemma. *Assume the cone σ is the sum $\tau + \varrho$ of a facet τ and a ray ϱ, with corresponding decompositions $V_\sigma \cong V_\tau \oplus V_\varrho$ and $A^\bullet_\sigma \cong B^\bullet_\tau \otimes_{\mathbf{R}} B^\bullet_\varrho$ as in 1.4.1. Then the minimal extension sheaf on $\langle\sigma\rangle$ satisfies $E^\bullet_\sigma \cong A^\bullet_\sigma \otimes_{B^\bullet_\tau} E^\bullet_\tau$. In particular, the restriction $E^\bullet_\sigma \longrightarrow E^\bullet_\tau$ induces an isomorphism $\overline{E}^\bullet_\sigma \cong \overline{E}^\bullet_\tau$ of graded vector spaces.*

Proof: We use induction on $\dim \sigma$. For $\gamma \prec \tau$ and $\hat\gamma := \gamma + \varrho$, the induction hypothesis yields that $E^\bullet_{\hat\gamma} \cong A^\bullet_{\hat\gamma} \otimes_{B^\bullet_\gamma} E^\bullet_\gamma$ with the algebra $B^\bullet_\gamma \subset A^\bullet_{\hat\gamma}$, the image of $S^\bullet(V^*_\gamma)$ in $A^\bullet_{\hat\gamma} = S^\bullet(V^*_{\hat\gamma})$ with respect to the map induced by the projection $V_{\hat\gamma} = V_\gamma \oplus V_\varrho \longrightarrow V_\gamma$.

Choosing a linear form $T \in A^2_\sigma$ vanishing on V_τ, we may write $A^\bullet_\sigma = B^\bullet_\tau[T]$, $A^\bullet_{\hat\gamma} = B^\bullet_\gamma[T]$ and thus $E^\bullet_{\hat\gamma} \cong A^\bullet_{\hat\gamma} \otimes_{B^\bullet_\gamma} E^\bullet_\gamma = E^\bullet_\gamma[T]$. Then there is an isomorphism $E^\bullet_{\partial\sigma} \cong E^\bullet_\tau \oplus T E^\bullet_{\partial\tau}[T]$ and it suffices to check that the restriction, which agrees with the natural map

$$A^\bullet_\sigma \otimes_{B^\bullet_\tau} E^\bullet_\tau \cong E^\bullet_\tau[T] = E^\bullet_\tau \oplus T E^\bullet_\tau[T] \longrightarrow E^\bullet_\tau \oplus T E^\bullet_{\partial\tau}[T]$$

induces an isomorphism mod $\mathbf{m}$. It is onto, since $E^\bullet_\tau \to E^\bullet_{\partial\tau}$ is. So the restriction mod $\mathbf{m}$ is so too, and it is into, since the composition $E^\bullet_\tau[T] \to E^\bullet_\tau \oplus T E^\bullet_{\partial\tau}[T] \to E^\bullet_\tau$ even is an isomorphism mod $\mathbf{m}$. □

If Δ is an N-rational fan for a lattice $N \subset V$ of rank $n = \dim V$, one associates to Δ a toric variety X_Δ with the action of the algebraic torus $\mathbf{T} := N \otimes_{\mathbf{Z}} \mathbf{C}^* \cong (\mathbf{C}^*)^n$. Let $IH^\bullet_{\mathbf{T}}(X_\Delta)$ denote the equivariant intersection cohomology of X_Δ with real coefficients. The following theorem, proved in [BBFK], has been the starting point to investigate minimal extension sheaves:

1.6 Theorem. *Let Δ be a rational fan and $\mathcal{E}^\bullet$ a minimal extension sheaf on Δ.*

i) The assignment

$$\mathcal{IH}^\bullet_{\mathbf{T}}\colon \Lambda \longmapsto IH^\bullet_{\mathbf{T}}(X_\Lambda)$$

defines a sheaf on the fan space Δ*, and that sheaf is a minimal extension sheaf.*

ii) For each cone $\sigma \in \Delta$*, the (non-equivariant) intersection cohomology sheaf* $\mathcal{IH}^\bullet$ *of* X_Δ *is constant along the corresponding* $\mathbf{T}$*-orbit with stalks isomorphic to* $\overline{E}_\sigma^\bullet$*.*

iii) If Δ *is complete or is affine of full dimension* n*, then one has*

$$IH^\bullet(X_\Delta) \;\cong\; \overline{E}_\Delta^\bullet \;.$$

Statement (iii) will be generalized in Theorem 3.8 to a considerably larger class of rational fans that we call "quasi-convex". – For a non-zero *rational* cone σ, the vanishing axiom for intersection cohomology together with statement (ii) yields $\overline{E}_\sigma^q = 0$ for $q \geq \dim\sigma$. This fact turns out to be a cornerstone in the recursive computation of intersection Betti numbers (see section 4). In the non-rational case, we have to state it as a condition; we conjecture that it holds in general:

1.7 Vanishing Condition $\mathbf{V}(\sigma)$**:** *A non-zero cone* σ *satisfies the condition* $\mathbf{V}(\sigma)$ *if*

$$\overline{E}_\sigma^q \;=\; 0 \quad \textit{for} \quad q \geq \dim\sigma \tag{1.7.1}$$

holds. A fan Δ *satisfies the condition* $\mathbf{V}(\Delta)$ *if* $\mathbf{V}(\sigma)$ *holds for each non-zero cone* $\sigma \in \Delta$*.*

We add some comments on that condition: The statements (ii) and (iii) in the following remark are not needed for later results; in particular, the results cited in their proof do not depend on these statements. – Statement (iii) has been influenced by a remark of Tom Braden.

1.8 Remark. i) If a fan Δ is simplicial or rational, then condition $\mathbf{V}(\Delta)$ is satisfied.

ii) Condition $\mathbf{V}(\sigma)$ is equivalent to

$$E^q_{(\sigma,\partial\sigma)} \;=\; \{0\} \quad \text{for} \quad q \leq \dim\sigma \;.$$

iii) If Δ satisfies $\mathbf{V}(\Delta)$, then every homomorphism $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ between minimal extension sheaves on Δ is determined by the homomorphism $\mathbf{R}^\bullet \cong E_o^\bullet \to F_o^\bullet \cong \mathbf{R}^\bullet$, see Proposition 1.3.

Proof: (i) The rational case has been mentioned above; for the simplicial case, see Proposition 1.4.

(ii) We may assume $\dim\sigma = n$; hence, the affine fan $\langle\sigma\rangle$ is "quasi-convex" (see Theorem 3.9). According to Corollary 5.6, there exists an isomorphism of abstract vector spaces $\overline{E}_\sigma^q \;\cong\; \overline{E}^{2n-q}_{(\sigma,\partial\sigma)}$. Hence $\overline{E}^q_{(\sigma,\partial\sigma)} \;=\; \{0\}$ for $q \leq \dim\sigma$, whence also $E^q_{(\sigma,\partial\sigma)} = \{0\}$ for $q \leq \dim\sigma$, since a homogeneous base of $\overline{E}^q_{(\sigma,\partial\sigma)}$ can be lifted to a homogeneous base of the free $A^\bullet$-module $E^q_{(\sigma,\partial\sigma)}$.

Now we may apply the following remark to the finitely generated $A^\bullet$-module $E^\bullet_{(\sigma,\partial\sigma)}$: For an $A^\bullet$-module $F^\bullet$ which is bounded from below one has either $F^\bullet = 0$

or, if r is minimal with $F^r \neq 0$, then $\overline{F}^r \cong F^r$ and $\overline{F}^q = 0$ for $q < r$. Thus the claim is immediate.

(iii) We use the terminology of the proof of Proposition 1.3: We have to show that a homomorphism $\varphi_{\partial\sigma}: E^\bullet_{\partial\sigma} \to F^\bullet_{\partial\sigma}$ extends in a unique way to a homomorphism $\varphi_\sigma: E^\bullet_\sigma \to F^\bullet_\sigma$. Statement (ii) implies that the restrictions $E^q_\sigma \to E^q_{\partial\sigma}$ and $F^q_\sigma \to F^q_{\partial\sigma}$ are isomorphisms for $q \leq \dim\sigma$. Since, as a consequence of $\mathbf{V}(\sigma)$, the $A^\bullet$-modules $E^\bullet_\sigma$ and $F^\bullet_\sigma$ can be generated by homogeneous elements of degree $< \dim\sigma$, the assertion follows. □

2. Combinatorial Pure Sheaves

In the case of a rational fan, "the" minimal extension sheaf is provided by the equivariant intersection cohomology sheaf (see Theorem 1.6). As in intersection cohomology, such a minimal extension sheaf may be embedded into a class of pure sheaves; its simple objects are generalizations of minimal extension sheaves. We introduce such sheaves and prove an analogue to the decomposition theorem in intersection cohomology.

2.1 Definition: *A (combinatorially)* ***pure*** *sheaf on a fan space Δ is a flabby sheaf $\mathcal{F}^\bullet$ of graded $\mathcal{A}^\bullet$-modules such that, for each cone $\sigma \in \Delta$, the $A^\bullet_\sigma$-module $F^\bullet_\sigma$ is finitely generated and free.*

2.2 Remark: As a consequence of the results in section 0.B and 0.C, we may replace flabbiness with the following "local" requirement: For each cone $\sigma \in \Delta$, the restriction homomorphism $\varrho^\sigma_{\partial\sigma}: F^\bullet_\sigma \to F^\bullet_{\partial\sigma}$ induces a surjective map $\overline{F}^\bullet_\sigma \to \overline{F}^\bullet_{\partial\sigma}$.

Pure sheaves are built up from simple objects whose prototypes are generalized minimal extension sheaves:

(Combinatorially) Simple Sheaves: For each cone $\sigma \in \Delta$, we construct inductively a "simple" sheaf ${}_\sigma\mathcal{E}^\bullet$ on Δ as follows: For a cone $\tau \in \Delta$ with $\dim\tau \leq \dim\sigma$, we set

$$ {}_\sigma E^\bullet_\tau \;:=\; {}_\sigma\mathcal{E}^\bullet(\tau) \;:=\; \begin{cases} A^\bullet_\sigma & \text{if } \tau = \sigma, \\ 0 & \text{otherwise.} \end{cases} $$

Now, if ${}_\sigma\mathcal{E}^\bullet$ has been defined on $\Delta^{\leq m}$ for some $m \geq \dim\sigma$, then for each $\tau \in \Delta^{m+1}$, we set

$$ {}_\sigma E^\bullet_\tau \;:=\; A^\bullet_\tau \otimes_{\mathbf{R}} {}_\sigma\overline{E}^\bullet_{\partial\tau} \;. $$

The restriction map $\varrho^\tau_{\partial\tau}: {}_\sigma E^\bullet_\tau \to {}_\sigma E^\bullet_{\partial\tau}$ is induced by some homogeneous $\mathbf{R}^\bullet$-linear section $s : {}_\sigma\overline{E}^\bullet_{\partial\tau} \to {}_\sigma E^\bullet_{\partial\tau}$ of the residue class map ${}_\sigma E^\bullet_{\partial\tau} \to {}_\sigma\overline{E}^\bullet_{\partial\tau}$.

Let us collect some useful facts about these sheaves.

2.2b Remark: i) The pure sheaf $\mathcal{F}^\bullet := {}_\sigma\mathcal{E}^\bullet$ is determined by the following properties of its reduction modulo $\mathbf{m}$:

a) $\overline{F}^\bullet_\sigma \cong \mathbf{R}^\bullet$,

b) for each cone $\tau \neq \sigma$, the reduced restriction map $\overline{F}^\bullet_\tau \to \overline{F}^\bullet_{\partial\tau}$ is an isomorphism.

ii) The sheaf ${}_\sigma\mathcal{E}^\bullet$ vanishes outside $\mathrm{st}_\Delta(\sigma)$ and can be obtained from a minimal extension sheaf ${}_{\Delta_\sigma}\mathcal{E}^\bullet$ on the transversal fan Δ_σ in the following way: Choose a decomposition $V = V_\sigma \oplus W$, and denote $B^\bullet \subset A^\bullet$ the image of $S^\bullet((V/V_\sigma)^*)$ in $A^\bullet$ and $B^\bullet_\sigma$ the image of $S^\bullet(V^*_\sigma)$ with respect to the projection with kernel W. Then $A^\bullet \cong B^\bullet_\sigma \otimes_{\mathbf{R}} B^\bullet$ and on $\mathrm{st}(\sigma)$ we have

$$ {}_\sigma\mathcal{E}^\bullet \cong B^\bullet_\sigma \otimes_{\mathbf{R}} ({}_{\Delta_\sigma}\mathcal{E}^\bullet) $$

where we identify Δ_σ with $\mathrm{st}(\sigma)$.

iii) For the zero cone o, the simple sheaf ${}_o\mathcal{E}^\bullet$ is the minimal extension sheaf of Δ.

iv) If Δ is a *rational* fan and $Y \subset X_\Delta$ the orbit closure associated to a cone $\sigma \in \Delta$, then the presheaf

$$ {}_Y\mathcal{IH}^\bullet_{\mathbf{T}} \;:\; \Lambda \;\mapsto\; IH^\bullet_{\mathbf{T}}(Y \cap X_\Lambda) $$

on Δ actually is a sheaf isomorphic to ${}_\sigma\mathcal{E}^\bullet$.

As main result of this section, we provide a Decomposition Formula for pure sheaves.

2.3 Algebraic Decomposition Theorem: *Every pure sheaf $\mathcal{F}^\bullet$ on Δ admits a direct sum decomposition*

$$ \mathcal{F}^\bullet \;\cong\; \bigoplus_{\sigma\in\Delta} \big({}_\sigma\mathcal{E}^\bullet \otimes_{\mathbf{R}} K^\bullet_\sigma\big) $$

with $K^\bullet_\sigma := K^\bullet_\sigma(\mathcal{F}^\bullet) := \ker\big(\overline{\varrho}^\sigma_{\partial\sigma}\colon \overline{F}^\bullet_\sigma \to \overline{F}^\bullet_{\partial\sigma}\big)$, a finite dimensional graded vector space.

Since a finite dimensional graded vector space $K^\bullet$ may be uniquely written in the form $K^\bullet = \bigoplus \mathbf{R}^\bullet[-\ell_i]^{n_i}$, we obtain the "classical" formulation

$$ \mathcal{F}^\bullet \cong \bigoplus_i {}_{\sigma_i}\mathcal{E}^\bullet[-\ell_i]^{n_i} $$

of the Decomposition Theorem.

Proof: The following result evidently allows an inductive construction of such a decomposition:

Given a pure sheaf $\mathcal{F}^\bullet$ and a cone σ of minimal dimension with $F^\bullet_\sigma \neq 0$, there is a decomposition $\mathcal{F}^\bullet = \mathcal{G}^\bullet \oplus \mathcal{H}^\bullet$ as a direct sum of pure $\mathcal{A}^\bullet$-submodules $\mathcal{G}^\bullet \cong_\sigma \mathcal{E}^\bullet \otimes_{\mathbf{R}} K^\bullet_\sigma$ and $\mathcal{H}^\bullet$, where $K^\bullet_\sigma = \overline{F}^\bullet_\sigma$.

Starting with $m = k := \dim\sigma$, we construct the decomposition recursively on each skeleton $\Delta^{\leq m}$: We set

$$\mathcal{G}^\bullet(\tau) := \begin{cases} F^\bullet_\sigma \cong A^\bullet_\sigma \otimes_{\mathbf{R}} K^\bullet_\sigma & \text{if } \tau = \sigma, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathcal{H}^\bullet(\tau) := \begin{cases} 0 & \text{if } \tau = \sigma, \\ \mathcal{F}^\bullet(\tau) & \text{otherwise.} \end{cases}$$

We now assume that for some $m \geq k$, we have constructed the decomposition on $\Delta^{\leq m}$. In order to extend it to $\Delta^{\leq m+1}$, it suffices again to extend it from the boundary fan $\partial\tau$ of some cone $\tau \in \Delta^{m+1}$ to the affine fan $\langle\tau\rangle$. By induction hypothesis, there exists a commutative diagram

$$\begin{tikzcd} & F^\bullet_\tau \arrow[r, two heads] \arrow[d] & F^\bullet_{\partial\tau} \arrow[r, phantom, "\cong"] \arrow[d] & G^\bullet_{\partial\tau} \oplus H^\bullet_{\partial\tau} \arrow[d] \\ K^\bullet_\tau \arrow[r, hook] & \overline{F}^\bullet_\tau \arrow[r, two heads] & \overline{F}^\bullet_{\partial\tau} \arrow[r, phantom, "\cong"] & \overline{G}^\bullet_{\partial\tau} \oplus \overline{H}^\bullet_{\partial\tau} \end{tikzcd} .$$

We now choose a first decomposition $\overline{F}^\bullet_\tau = K^\bullet_\tau \oplus L^\bullet \oplus M^\bullet$ satisfying $L^\bullet \cong \overline{G}^\bullet_{\partial\tau}$ and $M^\bullet \cong \overline{H}^\bullet_{\partial\tau}$. We may then lift it to a decomposition $F^\bullet_\tau = G^\bullet_\tau \oplus H^\bullet_\tau$ into free submodules such that $\overline{G}^\bullet_\tau = L^\bullet$ and $\overline{H}^\bullet_\tau = K^\bullet_\tau \oplus M^\bullet$ as well as $G^\bullet_\tau|_{\partial\tau} = G^\bullet_{\partial\tau}$ and $H^\bullet_\tau|_{\partial\tau} = H^\bullet_{\partial\tau}$. ◻

2.4 Geometric Decomposition Theorem: *Let $\pi : \check\Delta \to \Delta$ be a refinement map of fans with minimal extension sheaves $\check{\mathcal{E}}^\bullet$ and $\mathcal{E}^\bullet$, respectively. Then there is a decomposition*

$$\pi_*(\check{\mathcal{E}}^\bullet) \;\cong\; \mathcal{E}^\bullet \oplus \bigoplus_{\tau\in\Delta^{\geq 2}} {}_\tau\mathcal{E}^\bullet \otimes K^\bullet_\tau$$

of $\mathcal{A}^\bullet$-modules with cones $\tau \in \Delta^{\geq 2}$ and (positively) graded vector spaces $K^\bullet_\tau$.

Proof: For an application of the Algebraic Decomposition Theorem 2.3, we have to verify that the flabby sheaf $\pi_*(\check{\mathcal{E}}^\bullet)$ is pure. We still need to know that the $A^\bullet_\sigma$-modules $\pi_*(\check{\mathcal{E}}^\bullet)(\sigma)$ are free. If σ is an n-dimensional cone, then the affine fan $\langle\sigma\rangle$ is quasi-convex, see section 3. According to Corollary 3.11, the same holds true for the refinement $\check\sigma := \pi^{-1}(\langle\sigma\rangle) \preceq \check\Delta$. *Mutatis mutandis*, we may argue along the same lines for cones of positive codimension. – The fact that $\pi_*(\check{\mathcal{E}}^\bullet) \cong \mathcal{E}^\bullet \cong \mathcal{A}^\bullet$ on $\Delta^{\leq 1}$ provides the condition $\dim\tau \geq 2$, while $K^{<0}_\tau = 0$ is an obvious consequence of the corresponding fact for $\pi_*(\check{\mathcal{E}}^\bullet)$. ◻

2.5 Corollary: *Let $\pi\colon \check\Delta \to \Delta$ be a simplicial refinement of Δ. Then the minimal extension sheaf $\mathcal{E}^\bullet$ on Δ can be embedded as a direct factor into the sheaf of functions on $|\Delta|$ that are $\check\Delta'$-piecewise polynomial.*

Proof: According to Proposition 1.4, the sheaf $\check{\mathcal{A}}^\bullet$ is a minimal extension sheaf on $\check\Delta$. By Theorem 2.4, $\mathcal{E}^\bullet$ is a direct subsheaf of $\pi_*(\check{\mathcal{A}}^\bullet)$, which is the sheaf of functions on $|\Delta|$ that are $\check\Delta$-piecewise polynomial. ◻

3. Cellular Čech Cohomology of Minimal Extension Sheaves

In this section, our main aim is to characterize those fans Δ for which the $A^\bullet$-module $E^\bullet_\Delta$ of global sections of a minimal extension sheaf $\mathcal{E}^\bullet$ on Δ is free. The principal tool is a "cellular" cochain complex and the corresponding cohomology associated with a sheaf on a fan. The great interest in that freeness condition is due to the "Künneth formula" $E^\bullet_\Delta \cong A^\bullet \otimes_{\mathbf{R}^\bullet} \overline{E}^\bullet_\Delta$, which holds in that case. It allows us in sections 4 and 5 to compute virtual intersection Betti numbers and Poincaré duality first on the "equivariant" level $E^\bullet_\Delta$ and then to pass to "ordinary" (virtual) intersection cohomology $\overline{E}^\bullet_\Delta$.

The following name introduced for such fans is motivated by Theorem 3.9.

3.1 Definition: *A fan Δ is called **quasi-convex** if the $A^\bullet$-module $E^\bullet_\Delta$ is free.*

Obviously, quasi-convex fans are purely n-dimensional, i.e., each maximal cone in Δ is of dimension n. In the rational case, quasi-convexity can be reformulated in terms of the associated toric variety:

3.2 Theorem: *A rational fan Δ is quasi-convex if and only if the intersection cohomology of the associated toric variety X_Δ vanishes in odd degrees:*

$$IH^{\mathrm{odd}}(X_\Delta;\mathbf{R}) \;:=\; \bigoplus_{q\geq 0} IH^{2q+1}(X_\Delta;\mathbf{R}) \;=\; \{0\}\,.$$

In that case, there exists an isomorphism $IH^\bullet(X_\Delta) \cong \overline{E}^\bullet_\Delta$.

Proof: See Proposition 1.6 in [BBFK]. □

As the main tool to be used in the sequel, we now introduce the complex of cellular cochains on a fan with coefficients in a sheaf $\mathcal{F}$.

3.3 The cellular cochain complex. To a fan Δ and a sheaf $\mathcal{F}$ of real vector spaces on Δ, we associate its *cellular cochain complex* $C^\bullet(\Delta,\mathcal{F})$: The cochain modules are

$$C^k(\Delta,\mathcal{F}) := \bigoplus_{\dim\sigma=n-k} \mathcal{F}(\sigma) \quad \text{for} \quad 0\leq k\leq n\,. \tag{3.3.1}$$

To define the coboundary operator $\delta^k : C^k \to C^{k+1}$, we first fix, for each cone $\sigma\in\Delta$, an orientation $\mathrm{or}(\sigma)$ of V_σ such that $\mathrm{or}|_{\Delta^n}$ is constant. To each facet $\tau \prec_1 \sigma$, we then assign the orientation coefficient $\mathrm{or}^\sigma_\tau := 1$ if the orientation of V_τ, followed by the inward normal, coincides with the orientation of V_σ, and $\mathrm{or}^\sigma_\tau := -1$ otherwise. We then set

$$\delta(f)_\tau := \sum_{\sigma\succ_1\tau} \mathrm{or}^\sigma_\tau\; f_\sigma|_\tau \quad \text{for} \quad f=(f_\sigma)\in C^k(\Delta,\mathcal{F}) \quad \text{and} \quad \tau\in\Delta^{n-k-1}\,. \tag{3.3.2}$$

Up to a rearrangement of indices, the complex $C^\bullet(\Delta, \mathcal{E}^\bullet)$ for a minimal extension sheaf $\mathcal{E}^\bullet$ is a *minimal complex* in the sense of Bernstein and Lunts. We shall come back to that at the end of this section.

More generally, we also have to consider relative cellular cochain complexes with respect to a subfan.

3.4 Definition: *For a subfan Λ of Δ and a sheaf $\mathcal{F}$ of real vector spaces on Δ, we set*

$$C^\bullet(\Delta, \Lambda; \mathcal{F}) := C^\bullet(\Delta; \mathcal{F})/C^\bullet(\Lambda; \mathcal{F}) \quad \text{and} \quad H^q(\Delta, \Lambda; \mathcal{F}) := H^q(C^\bullet(\Delta, \Lambda; \mathcal{F}))$$

with the induced coboundary operator $\delta^\bullet := \delta^\bullet(\Delta, \Lambda; \mathcal{F})$. If Δ is purely n-dimensional and $\Lambda \preceq \partial\Delta$ a purely $n-1$-dimensional subfan with complementary subfan $\Lambda^* \preceq \partial\Delta$ (i.e. Λ^* is generated by the cones in $(\partial\Delta)^{n-1} \setminus \Lambda$), then the restriction of sections induces an augmented complex

$$\tilde{C}^\bullet(\Delta, \Lambda; \mathcal{F}): \qquad 0 \to F_{(\Delta,\Lambda^*)} \xrightarrow{\delta^{-1}} C^0(\Delta, \Lambda; \mathcal{F}) \xrightarrow{\delta^0} \ldots \longrightarrow C^n(\Delta, \Lambda; \mathcal{F}) \to 0$$

with cohomology $\tilde{H}^q(\Delta, \Lambda; \mathcal{F}) := H^q\big(\tilde{C}^\bullet(\Delta, \Lambda; \mathcal{F})\big)$.

In fact, we need only the two cases $\Lambda = \partial\Delta$ and $\Lambda = \emptyset$, where the complementary subfan is $\Lambda^* = \emptyset$ resp. $\Lambda^* = \partial\Delta$. We mainly are interested in the case where $\mathcal{F}$ is an $\mathcal{A}^\bullet$-module. Then, the cohomology $\tilde{H}^q(\Delta, \Lambda; \mathcal{F})$ is an $A^\bullet$-module. – In the augmented situation described above, we note that $C^0(\Lambda; \mathcal{F}) = 0$ and hence $C^0(\Delta, \Lambda; \mathcal{F}) = C^0(\Delta; \mathcal{F})$ holds.

We want to compare the above cohomology in the case of the constant sheaf $\mathcal{F} = \mathbf{R}$ with the usual real singular homology of a "spherical" cell complex associated with a purely n-dimensional fan Δ. To that end, we fix a euclidean norm on V and denote with $S_V \subset V$ its unit sphere. For a subfan Λ of Δ, we set

$$S_\Lambda := |\Lambda| \cap S_V \ .$$

For each non-zero cone σ in V, the subset $S_\sigma := \sigma \cap S_V$ is a closed cell of dimension $\dim \sigma - 1$. Hence, the collection $(S_\sigma)_{\sigma \in \Delta \setminus \{o\}}$ is a cell decomposition of S_Δ, and the corresponding (augmented) "homological" complex $C_\bullet(S_\Delta; \mathbf{R})$ of cellular chains with real coefficients essentially coincides with the cochain complex $C^\bullet(\Delta; \mathbf{R})$: We have $C^q(\Delta; \mathbf{R}) = C_{n-1-q}(S_\Delta; \mathbf{R})$ and $\delta^q = \partial_{n-1-q}$ for $q \le n-1$.

Let us call a *facet-connected component* of Δ each purely n-dimensional subfan Δ_0 being maximal with the property that every two n-dimensional cones $\sigma, \sigma' \in \Lambda$ can be joined by a chain $\sigma_0 = \sigma, \sigma_1, \ldots, \sigma_r = \sigma'$ of n-dimensional cones, where two consecutive ones meet in a facet.

3.5 Remark: Let Δ be a purely n-dimensional fan.

(i) If Δ is complete or $n \le 1$, then $\tilde{H}^\bullet(\Delta, \partial\Delta; \mathbf{R}) = 0$.

(ii) If Δ is not complete and $n \ge 2$, then

$$H^q(\Delta, \partial\Delta; \mathbf{R}) \;\cong\; H_{n-1-q}(S_\Delta, S_{\partial\Delta}; \mathbf{R}) \quad \text{for} \quad q > 0\;;$$

in particular, $H^q(\Delta, \partial\Delta; \mathbf{R}) = 0$ holds for $q \ge n-1$.

(iii) If s is the number of facet-connected components of Δ, then $\tilde{H}^0(\Delta, \partial\Delta; \mathbf{R}) \cong \mathbf{R}^{s-1}$.

Proof: The case $n \le 1$ is straightforward. For $n \ge 2$, the cohomology is computed via cellular homology; in the complete case, one has to use the fact that such a fan is facet-connected and that there is an isomorphism

$$\tilde{H}^q(\Delta; \mathbf{R}) \cong \tilde{H}_{n-1-q}(S_V; \mathbf{R}) \quad \text{for} \;\; n \ge 2 \;\; \text{and} \;\; q \ge 1\;. \tag{3.5.1}$$ □

In order to study the cellular cohomology of a flabby sheaf $\mathcal{F}$ of real vector spaces on Δ, we want to write such a sheaf as a direct sum of simpler sheaves: To a cone σ in Δ, we associate its *characteristic sheaf* ${}_\sigma\mathcal{J}$, i.e.,

$${}_\sigma\mathcal{J}(\Lambda) := \begin{cases} \mathbf{R} & \text{if } \sigma \in \Lambda \\ \{0\} & \text{otherwise} \end{cases} \;,$$

while the restriction homomorphisms are $id_{\mathbf{R}}$ or 0.

The following lemma is an elementary analogue of the “Algebraic Decomposition Theorem 2.3, and in fact has been motivated by it.

3.6 Lemma: *Every flabby sheaf $\mathcal{F}$ of real vector spaces on Δ admits a direct sum decomposition*

$$\mathcal{F} \;\cong\; \bigoplus_{\sigma\in\Delta} {}_\sigma\mathcal{J} \otimes K_\sigma$$

with real vector spaces $K_\sigma \;\cong\; \ker\bigl(\varrho^\sigma_{\partial\sigma}\colon \mathcal{F}(\sigma) \longrightarrow \mathcal{F}(\partial\sigma)\bigr)$.

This decomposition obviously is unique up to isomorphism.

Proof: The following arguments are analoguous to those in the proof of the Decomposition Theorem 2.3.

It clearly suffices to decompose such a flabby sheaf $\mathcal{F}$ as a direct sum

$$\mathcal{F} = \mathcal{G} \oplus \mathcal{H}$$

of flabby subsheaves $\mathcal{G}$ and $\mathcal{H}$ such that for some cone $\sigma \in \Delta$, we have $\mathcal{G} \cong {}_\sigma\mathcal{J} \otimes K_\sigma$ and $\mathcal{H}(\sigma) = 0$: Use induction over the number of cones $\tau \in \Delta$, such that $\mathcal{F}(\tau) \ne 0$.

Choose k minimal such that there is a k-dimensional cone σ with $\mathcal{F}(\sigma) \ne \{0\}$. Let $K_\sigma := \mathcal{F}(\sigma)$ and define the subsheaves $\mathcal{G}$ and $\mathcal{H}$ on the k-skeleton $\Delta^{\le k}$ as follows:

$$\mathcal{G}(\tau) := \begin{cases} K_\sigma & \text{, if } \tau = \sigma \\ 0 & \text{, otherwise} \end{cases}$$

while

$$\mathcal{H}(\tau) := \begin{cases} 0 & , \text{if } \tau = \sigma \\ \mathcal{F}(\tau) & , \text{otherwise} \end{cases} .$$

Now suppose that we already have constructed a decomposition $\mathcal{F} = \mathcal{G} \oplus \mathcal{H}$ on $\Delta^{\leq m}$ for some $m \geq k$. Let τ be a cone of dimension $m+1$. In particular, we have a decomposition

$$\mathcal{F}(\partial\tau) = \mathcal{G}(\partial\tau) \oplus \mathcal{H}(\partial\tau) .$$

Since $\mathcal{F}$ is flabby, the restriction map $\varrho^\tau_{\partial\tau}\colon \mathcal{F}(\tau) \to \mathcal{F}(\partial\tau)$ is surjective. We can find a decomposition $\mathcal{F}(\tau) = U \oplus W$ into complementary subspaces $U, W \subset \mathcal{F}(\tau)$ such that $\varrho^\tau_{\partial\tau}$ induces an isomorphism $U \stackrel{\cong}{\to} \mathcal{G}(\partial\tau)$ and an epimorphism $W \twoheadrightarrow \mathcal{H}(\partial\tau)$. Now set $\mathcal{G}(\tau) := U$ and $\mathcal{H}(\tau) := W$. In that manner, we can define $\mathcal{G}$ and $\mathcal{H}$ for all $(m+1)$-dimensional cones and thus on $\Delta^{\leq m+1}$. □

Since cellular cohomology commutes with direct sums and the tensor product with a fixed vector space, there is an isomorphism

$$\tilde{H}^\bullet(\Delta, \partial\Delta; \mathcal{F}) \cong \bigoplus_{\sigma\in\Delta} \tilde{H}^\bullet(\Delta, \partial\Delta; {}_\sigma\mathcal{J}) \otimes K_\sigma \ , \tag{3.6.1}$$

so it suffices to compute the cohomology of such a characteristic sheaf ${}_\sigma\mathcal{J}$.

3.7 Remark: For the characteristic sheaf ${}_\sigma\mathcal{J}$ of a cone $\sigma \in \Delta$, we have isomorphisms

$$\tilde{H}^\bullet(\Delta; {}_\sigma\mathcal{J}) \ \cong \ \tilde{H}^\bullet(\Delta_\sigma; \mathbf{R}) \quad \text{and} \quad \tilde{H}^\bullet(\Delta, \partial\Delta; {}_\sigma\mathcal{J}) \ \cong \ \tilde{H}^\bullet(\Delta_\sigma, \partial\Delta_\sigma; \mathbf{R}) .$$

with the transversal fan Δ_σ of $\sigma \in \Delta$. In particular, Remark 3.5 ii) implies

$$\tilde{H}^q(\Delta, \partial\Delta; {}_\sigma\mathcal{J}) = 0 \qquad \text{for} \quad q \ > \ n - \dim\sigma - 2$$

for every cone $\sigma \in \Delta$.

We are now ready to formulate the main result of this section.

3.8 Theorem (Characterization of Quasi-Convex Fans): *For a purely n-dimensional fan Δ and a minimal extension sheaf $\mathcal{E}^\bullet$ on it, the following statements are equivalent:*

i) For each cone $\sigma \in \Delta$, we have

$$\tilde{H}^\bullet(\Delta_\sigma, \partial\Delta_\sigma; \mathbf{R}) = 0 .$$

ii) We have

$$\tilde{H}^\bullet(\Delta, \partial\Delta; \mathcal{E}^\bullet) = 0 .$$

iii) The fan Δ is quasi-convex, i.e., the $A^\bullet$-module $E^\bullet_\Delta := \mathcal{E}^\bullet(\Delta)$ is free.

We put off the proof for a while, since we first want to deduce a *topological characterization* of quasi-convex fans.

3.9 Theorem: *A purely n-dimensional fan Δ is quasi-convex if and only if the support $|\partial\Delta|$ of its boundary fan is a real homology manifold. In particular, Δ is quasi-convex if Δ is complete or if S_Δ is a closed topological $(n-1)$-cell, e.g., if the support $|\Delta|$ or the complement of the support $V \setminus |\Delta|$ are convex sets.*

Proof: We note that condition (i) in Theorem 3.8 is satisfied for each cone σ of dimension $n-1$ or n, see 3.5 i), as well as for cones $\sigma \notin \partial\Delta$, since then Δ_σ is complete. In particular, that settles the case of a complete fan Δ. In the non-complete case, the proof is achieved by Proposition 3.10. □

To state the next result, we introduce this notation: For a cone σ in a fan Δ, we set $L_\sigma := S_{\Delta_\sigma} \subset (V/V_\sigma)$ and $\partial L_\sigma := S_{\partial\Delta_\sigma}$; in particular, we have $L_o = S_\Delta$. It is important to note that this cellular complex L_σ in the $(d-1)$-sphere S_{V/V_σ} (for $d := n - \dim\sigma$) may be identified with the *link* at an arbitrary point of the $(n-d-1)$-dimensional stratum $S_\sigma \setminus S_{\partial\sigma}$ of the stratified space S_Δ.

3.10 Proposition: *For a non-complete purely n-dimensional fan Δ, the following statements are equivalent:*

i) The fan Δ is quasi-convex.

ii) Each cone σ in $\partial\Delta$ satisfies the following condition:

(ii)$_\sigma$ The pair $(L_\sigma, \partial L_\sigma)$ is a real homology cell modulo boundary.

iii) Each cone σ in $\partial\Delta$ satisfies the following condition:

(iii)$_\sigma$ The link L_σ has the real homology of a point.

iv) Each cone σ in $\partial\Delta$ satisfies the following condition:

(iv)$_\sigma$ The boundary of the link ∂L_σ has the real homology of a sphere of dimension $n - \dim\sigma - 2$.

Proof: We first show that statement (ii) above and statement (i) of Theorem 3.8 are equivalent, thus reducing the equivalence "(i) $\Longleftrightarrow$ (ii)" to Theorem 3.8. As has been remarked above, it suffices to consider cones $\sigma \in (\partial\Delta)^{n-k}$ for $k \geq 2$. Part (ii) of Remark 3.5 implies that

$$H^q(\Delta_\sigma, \partial\Delta_\sigma; \mathbf{R}) \;\cong\; H_{k-1-q}(L_\sigma, \partial L_\sigma; \mathbf{R}) \qquad \text{for } q > 0\,. \tag{3.10.0}$$

For $q = 0$, we use the equivalence

$$H_{k-1}(L_\sigma, \partial L_\sigma; \mathbf{R}) \cong H^0(\Delta_\sigma, \partial\Delta_\sigma; \mathbf{R}) \cong \mathbf{R} \iff \tilde{H}^0(\Delta_\sigma, \partial\Delta_\sigma; \mathbf{R}) = 0\,.$$

In order to prove the equivalence of (ii), (iii), and (iv), we use induction on n. The case $n = 0$ is vacuous, and in case $n = 1$, it is trivial to check that (ii), (iii), and (iv) hold. We thus assume that the equivalence holds for every non-complete purely d-dimensional fan with $d \leq n-1$. If we apply that to the fans $\Delta_\sigma, \sigma \in \partial\Delta \setminus \{o\}$, we

see that the condition $ii)_\sigma$ is satisfied for every cone $\sigma \in \partial\Delta \setminus \{o\}$, if and only if $iii)_\sigma$ resp. $iv)_\sigma$ is. Hence it suffices to show the equivalence of $ii)_o, iii)_o$ and $iv)_o$ under that assumption. We need the following

Auxiliary Lemma: *Let $L := L_o$. If one of the conditions $ii)_\sigma, iii)_\sigma, iv)_\sigma$ is satisfied for every cone $\sigma \in \partial\Delta \setminus \{o\}$, the inclusion of the relative interior $\overset{\circ}{L} := L \setminus \partial L$ into L induces an isomorphism $H_\bullet(\overset{\circ}{L}) \cong H_\bullet(L)$, i.e., equivalently, the condition*

$$H_\bullet(L, \overset{\circ}{L}) = \{0\} \tag{3.10.1}$$

holds.

Proof. For $i = -1, \ldots, n-1$, we set $U_i := L \setminus (\partial L)_i$, where $(\partial L)_i$ is the i-skeleton of $\partial L = |\partial\Delta| \cap S_V$. By induction on i, we show that $H_\bullet(L, U_i) = 0$ holds. This is evident for $i = -1$, and the case $i = n-1$ is what we have to prove. For the induction step, we use the homology sequence associated to the triple (L, U_i, U_{i+1}) and show $H_\bullet(U_i, U_{i+1}) = 0$. Passing to a subdivision of barycentric type, we obtain an excision isomorphism between $H_\bullet(U_i, U_{i+1})$ and

$$H_\bullet\Big(\dot{\bigcup_{\sigma \in (\partial\Delta)^{i+2}}} \mathrm{st}'(\overset{\circ}{\sigma}), \dot{\bigcup_{\sigma \in (\partial\Delta)^{i+2}}} \big(\mathrm{st}'(\overset{\circ}{\sigma}) \setminus \overset{\circ}{\sigma}\big)\Big) = \bigoplus_{\sigma \in (\partial\Delta)^{i+2}} H_\bullet\big(\mathrm{st}'(\overset{\circ}{\sigma}), \mathrm{st}'(\overset{\circ}{\sigma}) \setminus \overset{\circ}{\sigma}\big)$$

where $\mathrm{st}'(\overset{\circ}{\sigma})$ denotes the open star of $\overset{\circ}{\sigma} \cap S_V$ with respect to that subdivision of $L = S_\Delta$. Furthermore, there is a homeomorphism $\mathrm{st}'(\overset{\circ}{\sigma}) \cong \overset{\circ}{c}(L_\sigma) \times (\overset{\circ}{\sigma} \cap S_V)$, where $\overset{\circ}{c}(L_\sigma)$ denotes the open cone over L_σ. By the Künneth formula, we thus obtain an isomorphism

$$H_\bullet\big(\mathrm{st}'(\overset{\circ}{\sigma}), \mathrm{st}'(\overset{\circ}{\sigma}) \setminus \overset{\circ}{\sigma}\big) \cong H_\bullet(\overset{\circ}{c}(L_\sigma), \overset{\circ}{c}(L_\sigma)^*) \cong \tilde{H}_\bullet(L_\sigma)[-1] = \{0\} \ .$$

since by the induction hypothesis $ii)_\sigma$ holds for every cone $\sigma \in \partial\Delta \setminus \{o\}$. □

"$ii)_o \iff iii)_o$" From the auxiliary lemma we obtain this chain of isomorphisms

$$H_q(L) \cong H_q(\overset{\circ}{L}) \cong H^{n-1-q}(S_V, S_V \setminus \overset{\circ}{L}) \cong H^{n-1-q}(L, \partial L) \cong H_{n-1-q}(L, \partial L)^* \ ,$$

where the first one follows from the above lemma, the second one, from topological (Poincaré-Alexander-Lefschetz) duality, the third one is obtained by excision, and the fourth one is the obvious duality. □

"$iii)_o \implies iv)_o$": We may assume $n \geq 3$ and have to show that ∂L has the same homology as an $(n-2)$-dimensional sphere. Using (iii) together with the equivalent assumption (ii), we have $\tilde{H}_{j-1}(\partial L) = \tilde{H}_j(L, \partial L) = \{0\}$ for $j \neq n-1$, and $H_{n-2}(\partial L) = \tilde{H}_{n-1}(L, \partial L) = \mathbf{R}$. Now apply the long exact homology sequence of the pair $(L, \partial L)$.

"$iv)_o \implies iii)_o$": It remains to verify that L has the homology of a point. We set $C := S^{n-1} \setminus \overset{\circ}{L}$ and look at the Mayer-Vietoris sequence

$$\ldots \to H_{q+1}(S^{n-1}) \to H_q(\partial L) \to H_q(L) \oplus H_q(C) \to H_q(S^{n-1}) \to H_{q-1}(\partial L) \to \ldots$$

associated to $S^{n-1} = L \cup C$. The hypothesis immediately yields $H_q(L) \oplus H_q(C) = 0$ for $1 \le q \le n-3$. The term $H_{n-1}(L) \oplus H_{n-1}(C)$ vanishes since both L and C are $(n-1)$-dimensional cell complexes in S^{n-1} with non-empty boundary. The following arrow $H_{n-1}(S^{n-1}) \to H_{n-2}(\partial L)$ is thus an isomorphism $\mathbf{R} \to \mathbf{R}$. This implies that the mapping $H_{n-2}(L) \oplus H_{n-2}(C) \to H_{n-2}(S^{n-1})$ is injective, too, and that yields $H_{n-2}(L) = 0$. For $q = 0$, we have a short exact sequence

$$0 \longrightarrow \mathbf{R} \longrightarrow H_0(L) \oplus H_0(C) \longrightarrow \mathbf{R} \longrightarrow 0 \ ,$$

and that yields the assertion. □

As a consequence, we see that quasi-convexity of a purely n-dimensional fan depends only on the topology of its boundary:

3.11 Corollary: *Let Δ and Δ' be purely n-dimensional fans. If their boundaries have the same support $|\partial\Delta| = |\partial\Delta'|$, then Δ is quasi-convex if and only if Δ' is.*

In particular, that applies to the following special cases:

i) *Δ' is a refinement of Δ,*

ii) *Δ and Δ' are "complementary" subfans, i.e., $\Delta \cup \Delta'$ is a complete fan, and Δ and Δ' have no n-dimensional cones in common.*

We now come to the proof of Theorem 3.8:

Proof of Theorem 3.8: For convenience, we briefly recall that we have to prove the equivalence of the following three statements for a purely n-dimensional fan Δ and the minimal extension sheaf $\mathcal{E}^\bullet$:

i) *For each cone $\sigma \in \Delta$, we have $\tilde{H}^\bullet(\Delta_\sigma, \partial\Delta_\sigma; \mathbf{R}) = 0$.*

ii) *We have $\tilde{H}^\bullet(\Delta, \partial\Delta; \mathcal{E}^\bullet) = 0$.*

iii) *The fan Δ is quasi-convex, i.e., the $A^\bullet$-module $E^\bullet_\Delta = \mathcal{E}^\bullet(\Delta)$ is free.*

"$(i) \iff (ii)$": If we write

$$\mathcal{E}^\bullet \cong \bigoplus_{\sigma\in\Delta} {}_\sigma\mathcal{J} \otimes K_\sigma$$

according to 3.6, we obtain the following direct sum decomposition

$$\tilde{H}^\bullet(\Delta, \partial\Delta; \mathcal{E}^\bullet) \cong \bigoplus_{\sigma\in\Delta} \tilde{H}^\bullet(\Delta_\sigma, \partial\Delta_\sigma; \mathbf{R}) \otimes K_\sigma$$

according to remark 3.7 and the isomorphism 3.6.1. Hence it is sufficient to see that none of the vector spaces $K_\sigma \cong \ker(\varrho^\sigma_{\partial\sigma} : E^\bullet_\sigma \longrightarrow E^\bullet_{\partial\sigma})$ is zero: Since $E^\bullet_\sigma$ is a non-zero free $A^\bullet_\sigma$-module and $E^\bullet_{\partial\sigma}$ is a torsion module, the restriction homomorphism $\varrho^\sigma_{\partial\sigma}$ never is injective.

"$(ii) \Longrightarrow (iii)$": We shall use the abbreviations

$$C^r := C^r(\Delta, \partial\Delta; \mathcal{E}^\bullet) \ , \ I^r := \operatorname{im} \delta^{r-1} \ , \quad \text{and} \quad \operatorname{Tor}_k := \operatorname{Tor}_k^{A^\bullet} \ .$$

By downward induction on r , we show the vanishing statement

$$\mathrm{Tor}_k(I^r, \mathbf{R}^\bullet) = 0 \text{ for } k > r \ .$$

In particular, the $A^\bullet$-module $I^0 = E^\bullet_\Delta$ satisfies $\mathrm{Tor}_1(I^0, \mathbf{R}^\bullet) = 0$ and thus is free according to (0.B).

Obviously the above statement holds for $r = n+1$. Since $C^\bullet$ is acyclic by assumption (ii), the sequences

$$0 \longrightarrow I^r \longrightarrow C^r \longrightarrow I^{r+1} \longrightarrow 0$$

are exact and thus induce exact sequences

$$\mathrm{Tor}_{k+1}(I^{r+1}, \mathbf{R}^\bullet) \longrightarrow \mathrm{Tor}_k(I^r, \mathbf{R}^\bullet) \longrightarrow \mathrm{Tor}_k(C^r, \mathbf{R}^\bullet) \ .$$

The last term vanishes for $k > r$: The module $C^r = \bigoplus_{\dim \sigma = n-r} E^\bullet_\sigma$ actually is a direct sum of shifted modules $A^\bullet_\sigma$, hence $\mathrm{Tor}_k(C^r, \mathbf{R}^\bullet) = 0$ for $k > r$, cf. 0.B.1. Since by induction hypothesis, also the first term vanishes, so does the second one.

"$(iii) \Longrightarrow (ii)$" In addition to the above, we use the abbreviations

$$K^r := \ker \delta^r \quad \text{and} \quad \tilde{H}^r := \tilde{H}^r(\Delta, \partial\Delta; \mathcal{E}^\bullet) = K^r / I^r \ .$$

We have to prove that $\tilde{H}^r = 0$ holds for all r. We choose an increasing sequence of subspaces $V_0 := 0 \subset V_1 \subset \ldots \subset V_n := V$ such that $V = V_r \oplus V_\sigma$ holds for all $\sigma \in \Delta^{n-r}$. Then the algebras $B^\bullet_r := S^\bullet((V/V_r)^*)$ form a decreasing sequence of subalgebras of $A^\bullet$, and for all cones $\sigma \in \Delta^{n-r}$, there is an isomorphism $B^\bullet_r \cong A^\bullet_\sigma$ induced from the composed mapping $V_\sigma \to V \to V/V_r$. In particular, each $C^r = \bigoplus_{\sigma \in \Delta^{n-r}} E^\bullet_\sigma$ is a free $B^\bullet_r$-module.

We now choose linear forms $T_1, \ldots, T_n \in A^2$ such that $B^\bullet_r = \mathbf{R}[T_1, \ldots, T_{n-r}]$. By induction on r, we shall prove:

$$\tilde{H}^q \ = \ 0 \quad \text{for} \quad q < r \ , \quad \text{and} \ \ I^r \ \text{ is a free } B^\bullet_r\text{-module.}$$

Since $I^0 = E^\bullet_\Delta$, the assertion is evident for $r = 0$. So let us proceed from r to $r+1$. The vanishing of $\tilde{H}^r$ is a consequence of the fact that its support in $\mathrm{Spec}(B^\bullet_r)$ is too small: According to Lemma 3.13 below, the support of $\tilde{H}^r$ in $\mathrm{Spec}(A^\bullet)$ has codimension at least $r+2$. Thus, as $B^\bullet_r$-module, its support in $\mathrm{Spec}(B^\bullet_r)$ has codimension at least 2. Using the exact sequence

$$0 \longrightarrow I^r \longrightarrow K^r \longrightarrow \tilde{H}^r \longrightarrow 0 \ ,$$

the vanishing $\tilde{H}^r = 0$ then follows from Lemma 3.12.

It remains to prove that $I := I^{r+1}$ is a free module over $B^\bullet := B^\bullet_{r+1}$. By 0.B, this is equivalent to

$$\mathrm{Tor}_1^{B^\bullet}(I, \mathbf{R}) = 0 \ . \tag{3.8.1}$$

Recall that $B_r^\bullet = B^\bullet[T]$ where $T := T_{n-r}$. Thus, the formula

$$\mathrm{Tor}_k^{B^\bullet}(I, \mathbf{R}) \cong \mathrm{Tor}_k^{B^\bullet[T]}(I, \mathbf{R}[T]) \tag{3.8.2}$$

provides the bridge to the induction hypothesis on the previous level r. Now the exact sequence

$$0 \longrightarrow \mathbf{R}[T] \xrightarrow{\mu} \mathbf{R}[T] \longrightarrow \mathbf{R} \longrightarrow 0\ . \tag{3.8.3}$$

where μ is multiplication with T, yields an exact sequence

$$\mathrm{Tor}_2^{B^\bullet[T]}(I, \mathbf{R}) \longrightarrow \mathrm{Tor}_1^{B^\bullet[T]}(I, \mathbf{R}[T]) \xrightarrow{\vartheta} \mathrm{Tor}_1^{B^\bullet[T]}(I, \mathbf{R}[T])\ . \tag{3.8.4}$$

The homomorphism ϑ is *injective*: The vector space $\mathrm{Tor}_2^{B^\bullet[T]}(I, \mathbf{R}) \cong \mathrm{Tor}_2^{B_r^\bullet}(I, \mathbf{R})$ vanishes since we already know that $\tilde{H}^r = \{0\}$ and thus the exact sequence

$$0 \longrightarrow I^r \longrightarrow C^r \longrightarrow I \longrightarrow 0 \tag{3.8.5}$$

is a resolution of I by *free* $B_r^\bullet$-modules.

Using the isomorphism (3.8.2), we may interpret ϑ as the $B^\bullet$-module homomorphism

$$\mathrm{Tor}_1^{B^\bullet}(\mu_I): \mathrm{Tor}_1^{B^\bullet}(I, \mathbf{R}) \longrightarrow \mathrm{Tor}_1^{B^\bullet}(I, \mathbf{R})$$

induced by $\mu_I: I \to I$, the multiplication with T. Now $\mathrm{Tor}_1^{B^\bullet}(I, \mathbf{R})$ is a finite dimensional graded vector space over $\mathbf{R}$. Hence, the injective endomorphism ϑ has to be surjective. On the other hand, μ_I and thus $\vartheta = \mathrm{Tor}_1^{B^\bullet}(\mu_I)$ has degree 2, so it is not surjective unless $\mathrm{Tor}_1^{B^\bullet}(I, \mathbf{R}) = 0$. This yields the desired vanishing result (3.8.1). ▫

We still have to state and prove the two lemmata referred to above. The first one is a general result of commutative algebra.

3.12 Lemma: *Let R be a polynomial algebra over a field and consider an exact sequence*

$$0 \longrightarrow R^s \longrightarrow M \longrightarrow L \longrightarrow 0$$

of R-modules. If M is torsion free and finitely generated, then either $L = 0$ or the codimension of its support $\mathrm{supp}(L)$ *in the spectrum* $\mathrm{Spec}\, R$ *is at most* 1.

Proof: We may assume that $Y := \mathrm{supp}\, L$ is a proper subset of $X := \mathrm{Spec}\, R$. Hence L is a torsion module, and thus M is of rank s. Let Q be the field of fractions of R. Since M is torsion-free, there is a natural monomorphism

$$M = M \otimes_R R \hookrightarrow M \otimes_R Q =: M_Q \cong Q^s\ .$$

We may interpret the given monomorphism $R^s \hookrightarrow M$ as an inclusion. Hence, an R-basis of R^s may be considered as a Q-basis of M_Q, thus providing an identification $M_Q = Q^s$.

We assume $\mathrm{codim}_X(Y) \geq 2$ and show $l = 0$ for every element $l \in L$. So fix an inverse image $m = (q_1, \ldots, q_s) \in M \subset Q^s$ of that element $l \in L$. A prime ideal $\mathbf{p}$

of R lies in $X \setminus Y$ if and only if the localized module $L_{\mathbf{p}}$ vanishes, or equivalently – since localization is exact –, if and only if the localized inclusion $(R_{\mathbf{p}})^s \hookrightarrow M_{\mathbf{p}}$ is an isomorphism. Hence, $\mathbf{p} \notin Y$ implies $q_1, \ldots, q_s \in R_{\mathbf{p}}$. Since a polynomial ring over a field is normal, the stipulation $\operatorname{codim}_X(Y) \geq 2$ yields $q_1, \ldots, q_s \in R$ and hence $m \in R^s$, thus proving $l = \{0\}$. □

3.13 Lemma: *The support of the $A^\bullet$-module $\tilde{H}^q(\Delta, \partial\Delta; \mathcal{E}^\bullet)$ has codimension $c \geq q + 2$ in* $\operatorname{Spec}(A^\bullet)$.

Proof: We show that the support is contained in the union of the "linear subspaces" $\operatorname{Spec}(A^\bullet_\sigma)$ of $\operatorname{Spec}(A^\bullet)$ with $\dim\sigma \leq n - q - 2$. To that end, we consider a prime ideal $\mathbf{p} \in \operatorname{Spec}(A^\bullet)$. Since localization of $A^\bullet$-modules at $\mathbf{p}$ is exact, the localized cohomology module $\tilde{H}^q_{\mathbf{p}}$ is the q-th cohomology of the complex

$$\tilde{C}^\bullet_{\mathbf{p}} \;\cong\; \tilde{C}^\bullet(\Delta, \partial\Delta; \mathcal{E}^\bullet_{\mathbf{p}}) \ ,$$

where the "localized" sheaf $\mathcal{E}^\bullet_{\mathbf{p}}$ is defined by setting

$$\mathcal{E}^\bullet_{\mathbf{p}}(\tau) := \mathcal{E}^\bullet(\tau)_{\mathbf{p}} \ .$$

Let k be the minimal dimension of a cone $\tau \in \Delta$ such that $\mathbf{p}$ belongs to $\operatorname{Spec}(A_\tau)$. Then $\mathcal{E}^\bullet_{\mathbf{p}}(\sigma) = 0$ for a cone with $\dim\sigma < k$, hence in particular

$$\mathcal{E}^\bullet_{\mathbf{p}} \cong \bigoplus_{\dim\sigma \geq k} {}_\sigma\mathcal{J} \otimes K_\sigma$$

with the characteristic sheaves ${}_\sigma\mathcal{J}$ and suitable vector spaces K_σ and thus, according to (3.6.1) and Remark 3.7

$$\tilde{H}^q(\Delta, \partial\Delta; \mathcal{E}^\bullet_{\mathbf{p}}) \cong \bigoplus_{\dim\sigma \geq k} \tilde{H}^q(\Delta, \partial\Delta; {}_\sigma\mathcal{J}) \otimes K_\sigma = 0 \text{ for } q > n - k - 2 \ .$$

Assume now $\tilde{H}^q(\Delta, \partial\Delta; \mathcal{E}^\bullet)_{\mathbf{p}} \cong \tilde{H}^q(\Delta, \partial\Delta; \mathcal{E}^\bullet_{\mathbf{p}}) \neq \{0\}$ for some $\mathbf{p}$ not contained in the union of the linear subspaces $\operatorname{Spec}(A^\bullet_\sigma)$ with $\dim\sigma \leq n - q - 2$. Thus we have $k > n - q - 2$ resp. $q > n - k - 2$, a contradiction. So $\operatorname{supp}(\tilde{H}^q(\Delta, \partial\Delta; \mathcal{E}^\bullet))$ is contained in the union of the "linear subspaces" $\operatorname{Spec}(A^\bullet_\sigma)$ with $\dim\sigma \leq n - q - 2$, as was to be proved. □

Eventually, we come back to the relation with the minimal complexes in the sense of Bernstein and Lunts: In [BeLu], a complex

$$K^\bullet : 0 \longrightarrow K^{-n} \stackrel{\delta^{-n}}{\longrightarrow} K^{-n+1} \stackrel{\delta^{-n+1}}{\longrightarrow} \ldots \stackrel{\delta^{-1}}{\longrightarrow} K^0 \longrightarrow 0$$

of graded $A^\bullet$-modules is called *minimal* if it satisfies the following conditions:

(i) $K^0 \cong \mathbf{R}^\bullet[n]$, i.e., the $A^\bullet$-module $A^\bullet/\mathbf{m} \cong \mathbf{R}^\bullet$ placed in degree $-n$;

(ii) there is a decomposition $K^{-d} = \bigoplus_{\sigma\in\Delta^d} K_\sigma$ for $0 \leq d \leq n$;

(iii) each K_σ is a free graded $A^\bullet_\sigma$-module;

(iv) for each cone $\sigma \in \Delta$, the differential δ maps K_σ to $\bigoplus_{\tau \prec_1 \sigma} K_\tau$, so for $\dim \sigma = d$, one obtains a subcomplex

$$0 \longrightarrow K_\sigma \xrightarrow{\delta_\sigma^{-d}} \bigoplus_{\tau \prec_1 \sigma} K_\tau \xrightarrow{\delta_\sigma^{-d+1}} \ldots \longrightarrow K_o \longrightarrow 0 \; ;$$

(v) with $I_\sigma := \ker \delta_\sigma^{-d+1}$, the differential δ_σ^{-d} induces an isomorphism

$$\overline{\delta}_\sigma^{-d} : \overline{K}_\sigma := K_\sigma / \mathbf{m} K_\sigma \xrightarrow{\cong} \overline{I}_\sigma := I_\sigma / \mathbf{m} I_\sigma$$

of real vector spaces.

If the fan Δ is purely n-dimensional, then the shifted cochain complex $K^\bullet := C^\bullet(\Delta, \mathcal{E}^\bullet[n])[n]$ – i.e., given by $K^{-i} := C^{n-i}(\Delta, \mathcal{E}^\bullet[n])$ – is minimal: With $K_\sigma := E^\bullet_\sigma[n]$, conditions (i) – (iv) are immediate; condition (v) follows from (LME) using the isomorphism $I_\sigma \cong \mathcal{E}^\bullet(\partial\sigma)[n] = E^\bullet_{\partial\sigma}[n]$ of $A^\bullet_\sigma$-modules.

Theorem 3.8 provides a characterization of quasi-convex fans in terms of acyclicity of the relative cellular cochain complex. An analoguous statement holds also for the absolute cellular cochain complex. In particular, this proves a conjecture of Bernstein and Lunts (see [BL], p.129, 15.9):

3.14 Theorem: *A purely n-dimensional fan Δ is quasi-convex if and only if the complex $C^\bullet(\Delta, \mathcal{E}^\bullet)$ is exact in degrees $q > 0$ and $H^0(\Delta, \mathcal{E}^\bullet) \cong E^\bullet_{(\Delta,\partial\Delta)}$. Specifically, for a complete fan Δ, a minimal complex in the sense of Bernstein and Lunts is exact.*

Proof: We consider the augmented absolute cellular cochain complex

$$0 \longrightarrow F_{(\Delta,\partial\Delta)} \longrightarrow C^0(\Delta; \mathcal{F}) \longrightarrow \ldots \longrightarrow C^n(\Delta; \mathcal{F}) \longrightarrow 0 \tag{3.14.1}$$

for some sheaf $\mathcal{F}$ on Δ. By 3.6.1, it is acyclic for the flabby sheaf $\mathcal{E}^\bullet$ if and only if it is acyclic for each characteristic sheaf ${}_\sigma\mathcal{J}$, where $\sigma \in \Delta$. For $\sigma \notin \partial\Delta$, that follows from 3.5.1 and Rem.3.5, since Δ_σ is complete. For a cone $\sigma \in \partial\Delta$, the absolute versions of Remark 3.7 and formula (3.10.0) yield isomorphisms

$$\tilde{H}^q(\Delta, {}_\sigma\mathcal{J}) \cong H^q(\Delta, {}_\sigma\mathcal{J}) \cong H^q(\Delta_\sigma, \mathbf{R}) \cong \tilde{H}_{k-1-q}(L_\sigma, \mathbf{R}) \; ,$$

where $n - k = \dim \sigma$ and L_σ is the link of some point $x \in S_\Delta \cap \mathring{\sigma}$. Now statemant iii) of Proposition 3.10 gives $\tilde{H}_{k-1-q}(L_\sigma, \mathbf{R}) = \{0\}$. □

For later use we still need the following result.

3.15 Corollary. *For a minimal extension sheaf $\mathcal{E}^\bullet$ on a quasi-convex fan Δ, the $A^\bullet$-submodule $E^\bullet_{(\Delta,\partial\Delta)} \subset E^\bullet_\Delta$ of global sections vanishing on the boundary fan $\partial\Delta$ is free.*

Proof: From the acyclicity of the absolute cellular cochain complex, we conclude as in the proof of Theorem 3.8 that $E^\bullet_{(\Delta,\partial\Delta)}$ is a free $A^\bullet$-module. □

4. Poincaré Polynomials

In the remaining part of our article, we want to discuss the virtual intersection Betti numbers $b_{2q}(\Delta) := \dim \overline{E}^{2q}_\Delta$ and $b_{2q}(\Delta, \partial\Delta) := \dim \overline{E}^{2q}_{(\Delta,\partial\Delta)}$ of a quasi-convex fan Δ, where $\mathcal{E}^\bullet$ is a minimal extension sheaf on Δ. It is convenient to use the language of Poincaré polynomials.

4.1 Definition: *The (equivariant) Poincaré series of a fan Δ is the formal power series*

$$Q_\Delta(t) := \sum_{q\geq 0} \dim\, E^{2q}_\Delta \cdot t^{2q} \ ,$$

its (intersection) Poincaré polynomial is the polynomial

$$P_\Delta(t) := \sum_{q\geq 0}^{<\infty} \dim\, \overline{E}^{2q}_\Delta \cdot t^{2q} = \sum_{q\geq 0}^{<\infty} b_{2q}(\Delta) t^{2q} \ .$$

For an affine fan $\langle\sigma\rangle$, we simply write

$$Q_\sigma := Q_{\langle\sigma\rangle} \ , \ P_\sigma := P_{\langle\sigma\rangle} \ .$$

Furthermore, for a subfan $\Lambda \preceq \Delta$, the relative Poincaré polynomial $P_{(\Delta,\Lambda)}$ is defined in an analoguous manner.

We refer to P_Δ as the *global* Poincaré polynomial of Δ, while the polynomials P_σ for $\sigma \in \Delta$ are called its *local* Poincaré polynomials.

4.2 Remark. *For a quasi-convex fan we have*

$$Q_\Delta(t) = \frac{1}{(1-t^2)^n} \cdot P_\Delta(t) \ ,$$

while for a cone σ, one has

$$Q_\sigma(t) = \frac{1}{(1-t^2)^{\dim \sigma}} \cdot P_\sigma(t) \ .$$

Proof. For a free graded $A^\bullet$-module $F^\bullet$, the Künneth formula $F^\bullet \cong A^\bullet \otimes_{\mathbf{R}} \overline{F}^\bullet$ holds, while the Poincaré series of a tensor product of graded vector spaces is the product of the Poincaré series of the factors. Since $Q_{A^\bullet} = 1/(1-t^2)^n$, the first formula follows immediately. Going over to the base ring $A^\bullet_\sigma$ yields the second one. □

The basic idea for the computation of the virtual intersection Betti numbers is to use a two-step procedure. In the first step, the global invariant is expressed as a

sum of local terms. In the second step, these local invariants are expressed in terms of the global ones associated to lower-dimensional fans.

4.3 Theorem (Local-to-Global Formula): If Δ is a quasi-convex fan of dimension n and $\overset{\circ}{\Delta}:= \Delta \setminus \partial\Delta$,, we have

$$P_\Delta(t) = \sum_{\sigma\in\overset{\circ}{\Delta}} (t^2-1)^{n-\dim\sigma} P_\sigma(t) \ ,$$

while

$$P_{(\Delta,\partial\Delta)}(t) = \sum_{\sigma\in\Delta} (t^2-1)^{n-\dim\sigma} P_\sigma(t) \ .$$

Proof. The cellular cochain complex

$$0 \longrightarrow E^\bullet_\Delta \longrightarrow C^0(\Delta,\partial\Delta;\mathcal{E}^\bullet) \longrightarrow \ldots \longrightarrow C^n(\Delta,\partial\Delta;\mathcal{E}^\bullet) \longrightarrow 0$$

of 3.4 associated to the quasi-convex fan Δ is acyclic by Theorem 3.8. We set

$$Q_i(t) := \sum_{q\ge 0} \dim C^i(\Delta,\partial\Delta;\mathcal{E}^{2q})\cdot t^{2q} = \sum_{\sigma\in\overset{\circ}{\Delta}\cap\Delta^{n-i}} Q_\sigma(t) \ .$$

Then we obtain the equality

$$Q_\Delta(t) = \sum_{i=0}^{n} (-1)^i Q_i(t) = \sum_{\sigma\in\overset{\circ}{\Delta}} (-1)^{n-\dim\sigma} Q_\sigma(t) \ .$$

The first assertion follows from Remark 4.2. The second formula is obtained in the same way using the acyclicity of the complex

$$0 \longrightarrow E^\bullet_{(\Delta,\partial\Delta)} \longrightarrow C^0(\Delta,\mathcal{E}^\bullet) \longrightarrow \ldots \longrightarrow C^n(\Delta,\mathcal{E}^\bullet) \longrightarrow 0$$

see Theorem 3.14 and Corollary 3.15. □

In order to reduce the computation of $\overline{E}^\bullet_\sigma \cong \overline{E}^\bullet_{\partial\sigma}$ to a problem in lower dimensions, we choose a line $L \subset V$ meetimg the relative interior $\overset{\circ}{\sigma}$ and consider the flattened boundary fan $\Lambda_\sigma := \pi(\partial\sigma)$, where $\pi : V_\sigma \to V_\sigma/L$ is the quotient map, cf. section 0.D. Then the direct image sheaf

(4.3.1) $$\mathcal{G}^\bullet := \pi_*(\mathcal{E}^\bullet|_{\partial\sigma}) : \ \tau \mapsto \mathcal{E}^\bullet\big((\pi|_{\partial\sigma})^{-1}(\tau)\big)$$

for $\tau\Lambda_\sigma$, is a minimal extension sheaf on Λ_σ. Writing $A^\bullet_\sigma = B^\bullet[T]$ with $B^\bullet := \pi^*(S^\bullet((V/L)^*)) \subset A^\bullet_\sigma$ and $T \in A^2_\sigma$ as in section 0.D, we obtain the identification

(4.3.2) $$\overline{E}^\bullet_\sigma \cong \overline{E}^\bullet_{\partial\sigma} \cong \overline{G}^\bullet_{\Lambda_\sigma}/(f\cdot\overline{G}^\bullet_{\Lambda_\sigma})$$

with the piecewise linear function $f := T\circ(\pi|_{\partial\sigma})^{-1} \in \mathcal{A}^2(\Lambda_\sigma)$. Here $E^\bullet_\sigma$ and $E^\bullet_{\partial\sigma}$ are considered as $A^\bullet_\sigma$-modules, while $G^\bullet_{\Lambda_\sigma}$ is a $B^\bullet$-module only; and it is with respect to that module structures one has to take residue class vector spaces.

As a first result we get an estimate for the degree of the Poincaré polynomials:

4.4 Corollary: *i) For a quasi-convex fan Δ the relative Poincaré polynomial $P_{(\Delta,\partial\Delta)}$ is monic of degree $2n$, whereas for a non-complete quasi-convex fan Δ the absolute Poincaré polynomial P_Δ is of degree at most $2n-2$.*

ii) For a non-zero cone σ, the "local" Poincaré polynomial P_σ is of degree at most $2\dim\sigma - 2$.

Proof. We proceed by induction on the dimension. Assuming that (ii) holds for every cone σ with $\dim\sigma \le n$, then Theorem 4.3 yields (i) in dimension $\dim\Delta = n$. Now, if σ is a ray, assertion (ii) is evident. For the induction step, we now assume $\dim\sigma = n > 1$. Going over to the complete fan Λ_σ of dimension $n-1$, we use the isomorphism (4.3.2). Since $\overline{G}^q_{\Lambda_\sigma} = 0$ holds for $q > 2n-2$ according to the induction hypothesis, assertion (ii) follows. □

For the second step, we have to relate the local Poincaré polynomial P_σ to the global Poincaré polynomial $P_{\Lambda_\sigma}(t)$ of the complete (and thus quasi-convex) fan Λ_σ of dimension $\dim\sigma - 1$. Here the vanishing condition $V(\sigma)$, cf.1.7, plays a decisive role:

4.5 Theorem (Local Recursion Formula): *Let σ be a cone.*

i) If σ is simplicial, then we have $P_\sigma \equiv 1$.

ii) If the condition $V(\sigma)$ is satisfied and σ is not the zero cone, then we have

$$P_\sigma(t) = \tau_{<\dim\sigma}\big((1-t^2)P_{\Lambda_\sigma}(t)\big) .$$

In the statement above, the truncation operator $\tau_{<k}$ is defined by $\tau_{<k}(\sum_q a_q t^q) := \sum_{q<k} a_q t^q$. – Let us note that for $\dim\sigma = 1$ and 2, the statements (i) and (ii) agree.

Proof: Statement i) follows from the fact that $E^\bullet_\sigma \cong A^\bullet_\sigma$ for a simplicial cone σ. In order to prove statement ii) we use the isomorphism (4.3.2), thus have to investigate the graded vector space $\overline{G}^\bullet_{\Lambda_\sigma}/f\overline{G}^\bullet_{\Lambda_\sigma}$ respectively the kernel and cokernel of the map

$$\overline{\mu} : \overline{G}^\bullet_{\Lambda_\sigma}[-2] \longrightarrow \overline{G}^\bullet_{\Lambda_\sigma}, \overline{h} \mapsto \overline{fh}$$

induced by the multiplication $\mu\colon G^\bullet_{\Lambda_\sigma}[-2] \to G^\bullet_{\Lambda_\sigma}$ with the piecewise linear function $f \in \mathcal{A}^2(\Lambda_\sigma)$. We apply the following "Hard Lefschetz" type theorem with $\Delta = \Lambda_\sigma$ and $\mathcal{E}^\bullet = \mathcal{G}^\bullet$:

4.6 Theorem: *Let Δ be a complete fan and $f \in \mathcal{A}^2(\Delta)$ a strictly convex function, i.e., $f = (f_\sigma)_{\sigma\in\Delta}$ is convex and $f_\sigma \ne f_{\sigma'}$ for $\sigma \ne \sigma'$, furthermore let $\gamma^+(f) \subset V \times \mathbf{R}$*

be the convex hull of the graph $\Gamma_f \subset V \times \mathbf{R}$ of f. Then, if the condition $V\big(\gamma^+(f)\big)$ is satisfied, the map

$$\overline{\mu}^{2q} : \overline{E}^{2q}_{\Delta} \longrightarrow \overline{E}^{2q+2}_{\Delta}$$

induced by the multiplication $\mu : E^\bullet_\Delta[-2] \to E^\bullet_\Delta$, $h \mapsto fh$, is injective for $2q \le n-1$ and surjective for $2q \ge n-1$.

Theorem 4.6 will be derived from the vanishing condition $V(\gamma^+(f))$ at the end of section 5 by means of Poincaré duality. As a simple consequence of Corollary 5.6, a "numerical" version of Poincaré duality can be formulated as follows.

4.7 Theorem: *For a quasi-convex fan Δ, the global Poincaré polynomials P_Δ and $P_{(\Delta,\partial\Delta)}$ satisfy the identity*

$$P_{(\Delta,\partial\Delta)}(t) = t^{2n} P_\Delta(t^{-1}) \ .$$

We conclude this section with an application of the decomposition theorem 2.3, which has been communicated to us by Tom Braden (cf. also [BrMPh]):

4.8 Theorem (Kalai's conjecture) *For a face $\tau \preceq \sigma$ of the cone σ with transversal fan Δ_τ in $\Delta := \langle\sigma\rangle$ we have:*

$$P_\sigma(t) \ge P_\tau(t) \cdot P_{\Delta_\tau}(t) \ ,$$

where $P \ge Q$ means, that the corresponding inequality holds for the coefficients of monomials of P and Q with the same degree.

Proof. Consider the minimal extension sheaf $\mathcal{E}^\bullet$ on the affine fan $\Delta := \langle\sigma\rangle$ and denote $\mathcal{F}^\bullet$ the trivial extension of $\mathcal{E}^\bullet|_{\mathrm{st}(\tau)}$, i.e. if $\Lambda \preceq \Delta$ and Λ_0 is generated by the cones in $\Lambda \cap \mathrm{st}(\tau)$, then $\mathcal{F}^\bullet(\Lambda) = \mathcal{E}^\bullet(\Lambda_0)$. Obviously $\mathcal{F}^\bullet$ is a pure sheaf and its decomposition has the form

$$\mathcal{F}^\bullet \cong ({}_\tau\mathcal{E}^\bullet \otimes \overline{E}^\bullet_\tau) \oplus \bigoplus_{\gamma \succ \tau} {}_\gamma\mathcal{E}^\bullet \otimes K^\bullet_\gamma \ .$$

Now our inequality follows immediately by taking the residue class module of the global sections of the above sheaf, since ${}_\tau\mathcal{E}^\bullet \cong B^\bullet_\tau \otimes_{\mathbf{R}} ({}_{\Delta_\tau}\mathcal{E}^\bullet)$ (identifying Δ_τ with $\mathrm{st}(\tau)$) with $B^\bullet := S((V/V_\tau)^*) \subset A^\bullet$ and $B^\bullet_\tau \subset A^\bullet$, such that $B^\bullet_\tau \cong A^\bullet_\tau$, hence in particular $A^\bullet \cong B^\bullet_\tau \otimes_{\mathbf{R}} B^\bullet$. □

5. Poincaré Duality

In this section, we first define a – non-canonical – "intersection product" $\mathcal{E}^\bullet \times \mathcal{E}^\bullet \to \mathcal{E}^\bullet$ on a minimal extension sheaf $\mathcal{E}^\bullet$ for an arbitrary fan Δ. On the level of global sections, it provides a "product" $E^\bullet_\Delta \times E^\bullet_{(\Delta,\partial\Delta)} \to E^\bullet_{(\Delta,\partial\Delta)}$. If the fan is even quasi-convex, then in addition, there exists an evaluation mapping $\varepsilon : E^\bullet_{(\Delta,\partial\Delta)} \to A^\bullet[-2n]$. The main result of this section is the "Poincaré Duality Theorem" 5.3 according to which the composition of the intersection product and the evaluation map is a dual pairing.

In the case of a simplicial fan, where the sheaf $\mathcal{A}^\bullet$ of piecewise polynomial functions is a minimal extension sheaf, such a product is simply given by the multiplication of these functions. Hence, one possible approach to the general case is as follows: Choose a simplicial refinement $\hat{\Delta}$ of Δ and, according to the Decomposition Theorem 2.4, interpret $\mathcal{E}^\bullet$ as a direct factor of the sheaf $\hat{\mathcal{A}}$ of $\hat{\Delta}$-piecewise polynomial functions on Δ. Then take the restriction of the multiplication of functions on $\hat{\mathcal{A}}^\bullet$ to $\mathcal{E}^\bullet \subset \hat{\mathcal{A}}^\bullet$ and project to $\mathcal{E}^\bullet$.

But in order to keep track of the relation between the intersection product over the boundary of a cone and the cone itself, it is useful to apply the above idea repeatedly in a recursive extension procedure. The proof of Poincaré duality will follow the same pattern.

5.1 An Intersection Product: The 2-dimensional skeleton $\Delta^{\leq 2}$ is a simplicial subfan. Hence, up to a scalar multiple, there is a canonical isomorphism $\mathcal{A}^\bullet \cong \mathcal{E}^\bullet$ on $\Delta^{\leq 2}$ (see 1.8). We thus define the intersection product on $\Delta^{\leq 2}$ to correspond via that isomorphism to the product of functions.

We now assume that the intersection product is defined on $\Delta^{\leq m}$ and consider a cone $\sigma \in \Delta^{m+1}$. So we are given a symmetric bilinear morphism $E^\bullet_{\partial\sigma} \times E^\bullet_{\partial\sigma} \to E^\bullet_{\partial\sigma}$ of $A^\bullet_\sigma$-modules. As in the previous section, let $L \subset V_\sigma$ be a line intersecting $\overset{\circ}{\sigma}$ and let $B^\bullet \subset A^\bullet_\sigma$ be the image of $S^\bullet((V_\sigma/L)^*)$ in $A^\bullet_\sigma = S(V^*_\sigma)$. We recall that $E^\bullet_{\partial\sigma} \cong G^\bullet_{\Lambda_\sigma}$ (see the remarks preceding Corollary 4.4) is a free $B^\bullet$-module, by Theorem 3.8. Let us define the $A^\bullet_\sigma$-module

$$F^\bullet_\sigma \;:=\; A^\bullet_\sigma \otimes_{B^\bullet} E^\bullet_{\partial\sigma}\ . \tag{5.1.1}$$

Since $E^\bullet_\sigma$ is a free $A^\bullet_\sigma$ -module, the (surjective) restriction $E^\bullet_\sigma \to E^\bullet_{\partial\sigma}$ can be factorized

$$E^\bullet_\sigma \xrightarrow{\alpha} A^\bullet_\sigma \otimes_{B^\bullet} E^\bullet_{\partial\sigma} = F^\bullet_\sigma \xrightarrow{\beta} A^\bullet_\sigma \otimes_{A^\bullet_\sigma} E^\bullet_{\partial\sigma} = E^\bullet_{\partial\sigma}\ .$$

The map $\alpha\colon E^\bullet_\sigma \to F^\bullet_\sigma$ is a "direct" embedding, i.e., there is a decomposition $F^\bullet_\sigma \cong \alpha(E^\bullet_\sigma) \oplus K^\bullet$, since the reduction of α mod $\mathbf{m} \subset A^\bullet_\sigma$ is injective and $F^\bullet_\sigma$ a free $A^\bullet_\sigma$-module. We may even assume that $K^\bullet$ is contained in the kernel of the natural map $A^\bullet_\sigma \otimes_{B^\bullet} E^\bullet_{\partial\sigma} \to E^\bullet_{\partial\sigma}$: Take a homogeneous basis $f_1, \ldots, f_r$ of $K^\bullet$. The images $\beta(f_i)$

of these elements in $E^\bullet_{\partial\sigma}$ are restrictions of elements $g_i \in E^\bullet_\sigma$; hence, we may replace $K^\bullet$ by the submodule generated by the elements $f_i - \alpha(g_i)$ for $1 \le i \le r$.

On the other hand, by scalar extension, there is an induced product

$$F^\bullet_\sigma \times F^\bullet_\sigma \longrightarrow F^\bullet_\sigma \ .$$

It provides the desired extension of the intersection product from $\partial\sigma$ to σ via the composition

$$E^\bullet_\sigma \times E^\bullet_\sigma \xrightarrow{\alpha\times\alpha} F^\bullet_\sigma \times F^\bullet_\sigma \longrightarrow F^\bullet_\sigma = \alpha(E^\bullet_\sigma) \oplus K^\bullet \longrightarrow \alpha(E^\bullet_\sigma) \cong E^\bullet_\sigma \ ,$$

where the last arrow is the projection onto $\alpha(E^\bullet_\sigma)$ with kernel $K^\bullet$. This ends the extension procedure. To sum up, after a finite number of steps, we arrive at a symmetric bilinear morphism

$$\mathcal{E}^\bullet \times \mathcal{E}^\bullet \longrightarrow \mathcal{E}^\bullet \tag{5.1.2}$$

of sheaves of $\mathcal{A}^\bullet$-modules, called an *intersection product on the minimal extension sheaf* $\mathcal{E}^\bullet$. In particular, we thus have defined a product

$$E^\bullet_\Delta \times E^\bullet_\Delta \longrightarrow E^\bullet_\Delta \ ,$$

and obviously we have $E^\bullet_\Delta \cdot E^\bullet_{(\Delta,\partial\Delta)} \subset E^\bullet_{(\Delta,\partial\Delta)}$.

In order to obtain a dual pairing in the case of a quasi-convex fan Δ, we compose the induced product $E^\bullet_\Delta \times E^\bullet_{(\Delta,\partial\Delta)} \to E^\bullet_{(\Delta,\partial\Delta)}$ with an "evaluation" homomorphism

$$\varepsilon : E^\bullet_{(\Delta,\partial\Delta)} \longrightarrow A^\bullet[-2n]$$

that can be defined as follows: As a consequence of Corollary 4.4, we know

$$\overline{E}^q_{(\Delta,\partial\Delta)} = \begin{cases} \mathbf{R} & , q = 2n \\ 0 & , q > 2n \end{cases} \ .$$

Moreover, according to Corollary 3.15, $E^\bullet_{(\Delta,\partial\Delta)}$ is a free $A^\bullet$-module. Hence, there is an element $\varepsilon \in \mathrm{Hom}_{A^\bullet}(E^\bullet_{(\Delta,\partial\Delta)}, A^\bullet[-2n]) \setminus \{0\}$ of degree zero; in fact it is unique up to multiplication by a real scalar. If Δ is a simplicial fan, this homomorphism ε can be described quite explicitly: Following [Bri, p.13], one fixes some euclidean norm on V and hence also on V^*. Write each cone $\sigma \in \Delta^n$ as $\sigma = H_1 \cap \ldots \cap H_n$ with half spaces $H_i = H_{\alpha_i}$ and linear forms $\alpha_i \in V^*$ of length 1, finally set $f_\sigma := \alpha_1 \cdot \ldots \cdot \alpha_n$. Then the map ε is of the following form:

$$E^\bullet_{(\Delta,\partial\Delta)} \cong A^\bullet_{(\Delta,\partial\Delta)} \subset \bigoplus_{\sigma\in\Delta^n} A^\bullet_\sigma \longrightarrow A^\bullet[-2n] \ , \ \ h = (h_\sigma)_{\sigma\in\Delta^n} \mapsto \sum_{\sigma\in\Delta^n} \frac{h_\sigma}{f_\sigma} \ .$$

For example, if $\Delta = \langle\sigma\rangle$ is a full-dimensional affine simplicial fan, then $A^\bullet_{(\sigma,\partial\sigma)}$ is of the form $f_\sigma A^\bullet_\sigma$ for the function $f_\sigma \in A^{2n}$ as above, and hence the above map has values in $A^\bullet \subset Q(A^\bullet)$. In the general case for each $n-1$-cone in $\overset{\circ}{\Delta}$ there are two

summands which have a pole of order 1 along it, but these poles cancel one another, while each summand already is regular along $\partial\Delta$ because of $h \in A^\bullet_{(\Delta,\partial\Delta)}$.

Since the intersection product $\mathcal{E}^\bullet \times \mathcal{E}^\bullet \to \mathcal{E}^\bullet$ is a homomorphism of sheaves, we may sum up the general situation as follows: For a quasi-convex fan Δ, there exists *homogenous pairings* (i.e., of degree zero with respect to the total grading on the product)

$$E^\bullet_\Delta \times E^\bullet_{(\Delta,\partial\Delta)} \longrightarrow E^\bullet_{(\Delta,\partial\Delta)} \longrightarrow A^\bullet[-2n] \tag{5.1.3}$$

and

$$\overline{E}^\bullet_\Delta \times \overline{E}^\bullet_{(\Delta,\partial\Delta)} \longrightarrow \overline{E}^\bullet_{(\Delta,\partial\Delta)} \longrightarrow \mathbf{R}^\bullet[-2n]\ . \tag{5.1.4}$$

Our aim is to prove that these are in fact both dual pairings. Fortunately, it suffices to verify that property for one of these two pairings: According to the very definition of a minimal extension sheaf and by Theorem 3.8 and Corollary 3.15, the $A^\bullet$-modules $E^\bullet_\Delta$ and $E^\bullet_{(\Delta,\partial\Delta)}$ are free. We may thus apply the following result.

5.2 Lemma. *Let $E^\bullet$, $F^\bullet$ be two finitely generated free graded $A^\bullet$-modules. Then a homogeneous pairing*

$$E^\bullet \times F^\bullet \to A^\bullet[r]$$

is dual if and only if the induced pairing

$$\overline{E}^\bullet \times \overline{F}^\bullet \longrightarrow \overline{A}^\bullet[r]$$

is.

Proof. After shifting the grading of $F^\bullet$, we may assume $r = 0$. We choose homogeneous bases of $E^\bullet$ and $F^\bullet$, so the pairing can be represented by a matrix M over $A^\bullet$. Then M is a square matrix and is invertible if and only if that holds for its residue class mod $\mathbf{m}_A$: The implication "$\Longrightarrow$" is obvious, while for "$\Longleftarrow$", it suffices to prove $\det M \in A^0 = \mathbf{R}$.

We arrange the bases in increasing order for $E^\bullet$ and decreasing order for $F^\bullet$ with respect to the degrees. Since the induced pairing is a dual one, the submodules of $E^\bullet$ and $F^\bullet$ generated by basis elements of opposite degrees have the same rank. The matrix of the pairing is composed of square matrices with entries in A^0 along the diagonal, and below these all entries are 0. Thus the determinant $\det M$ equals the product of the determinants of the diagonal square blocks, so it is a constant. ◻

We come now to the central result of this section:

5.3 Theorem (Poincaré Duality (PD)): *For a quasi-convex fan Δ of dimension n, the composition*

$$E^\bullet_\Delta \times E^\bullet_{(\Delta,\partial\Delta)} \longrightarrow E^\bullet_{(\Delta,\partial\Delta)} \longrightarrow A^\bullet[-2n]$$

is a dual pairing of finitely generated free $A^\bullet$-modules.

Proof: For an affine fan $\Delta := \langle\sigma\rangle$ with a cone of dimension $\dim\sigma = n \le 2$ Poincaré duality obviously holds. Now Theorem 5.3 follows by the following two lemmata, since using them one can prove the general case with a two step induction procedure.

5.3a Lemma. *If Poincaré duality holds for complete fans in dimensions $< n$, then also for an affine fan $\Delta = \langle\sigma\rangle$ with a cone σ of dimension n.*

Proof. According to (4.3.1), we identify $E^\bullet_{\partial\sigma}$ with the $B^\bullet$-module $G^\bullet_{\Lambda_\sigma}$ of global sections of a minimal extension sheaf $\mathcal{G}^\bullet$ on the fan Λ_σ in V/L. Since the fan Λ_σ is of dimension $< n$, we obtain a dual pairing

$$E^\bullet_{\partial\sigma} \times E^\bullet_{\partial\sigma} \longrightarrow E^\bullet_{\partial\sigma} \longrightarrow B^\bullet[2-2n]\ .$$

By extension of scalars, that induces dual pairings

$$F^\bullet_\sigma \times F^\bullet_\sigma \longrightarrow F^\bullet_\sigma \xrightarrow{\ \eta\ } A^\bullet[2-2n]$$

resp.

$$\overline{F}^\bullet_\sigma \times \overline{F}^\bullet_\sigma \longrightarrow \overline{F}^\bullet_\sigma \longrightarrow \mathbf{R}^\bullet[2-2n]$$

and, after a shift,

$$\overline{F}^\bullet_\sigma \times \overline{F}^\bullet_\sigma[-2] \longrightarrow \overline{F}^\bullet_\sigma[-2] \longrightarrow \mathbf{R}^\bullet[-2n]\ .$$

To achieve the proof, we are going to construct a factorization of the induced pairing $\overline{E}^\bullet_\sigma \times \overline{E}^\bullet_{(\sigma,\partial\sigma)} \to \mathbf{R}^\bullet[-2n]$ on the level of residue class vector spaces in the following form:

$$\overline{E}^\bullet_\sigma \times \overline{E}^\bullet_{(\sigma,\partial\sigma)} \xrightarrow{\alpha\times\vartheta} \overline{F}^\bullet_\sigma \times \overline{F}^\bullet_\sigma[-2] \longrightarrow \overline{F}^\bullet_\sigma[-2] \longrightarrow \mathbf{R}^\bullet[-2n]\,. \tag{5.3.1}$$

We show the existence of a homomorphism $\mu : \overline{F}^\bullet_\sigma[-2] \to \overline{F}^\bullet_\sigma$ such that α and ϑ induce isomorphisms

$$\overline{E}^\bullet_{(\sigma,\partial\sigma)} \cong \ker\mu \qquad \text{and} \qquad \overline{E}^\bullet_\sigma \cong \operatorname{coker}\mu\ .$$

Finally, forgetting about the shifts, the map μ is self-adjoint with respect to the above dual pairing on $\overline{F}^\bullet_\sigma$. Hence, the restriction of the pairing to $\ker\mu \times \operatorname{coker}\mu$ is dual, too.

We interpret $F^\bullet_\sigma$ as the module of sections of a sheaf of $\mathcal{A}^\bullet$-modules on the affine fan $\langle\sigma\rangle$. To that end, we consider the subdivision

$$\Sigma := \partial\sigma \cup \{\hat\tau := \tau + \varrho; \tau \in \partial\sigma\}\ .$$

of $\langle\sigma\rangle$, where ϱ is the ray $L \cap \sigma$. Let $B^\bullet_\tau \subset A^\bullet_{\hat\tau}$ denote the subalgebra of functions constant on parallels to the line L. Then, according to remark 1.5, the sheaf $\mathcal{F}^\bullet$ on Σ with

$$\tau \mapsto F^\bullet_\tau := E^\bullet_\tau\ ,\ \hat\tau \mapsto F^\bullet_{\hat\tau} := A^\bullet_{\hat\tau} \otimes_{B^\bullet_\tau} E^\bullet_\tau \qquad \text{for } \tau \in \partial\sigma$$

and the obvious restriction homomorphisms is a minimal extension sheaf $\mathcal{F}^\bullet$ on Σ and satisfies $\mathcal{F}^\bullet(\Sigma) \cong A^\bullet \otimes_{B^\bullet} E^\bullet_{\partial\sigma} = F^\bullet_\sigma$. Furthermore, the sheaf $\mathcal{F}^\bullet$ inherits an intersection product from $\mathcal{E}^\bullet|_{\partial\sigma} \cong \mathcal{F}^\bullet|_{\partial\sigma}$ as in 5.1.

For simplicity, we interpret the mapping α in 5.1 as an inclusion $E^\bullet_\sigma \subset F^\bullet_\sigma$ and identify $\mathcal{F}^\bullet$ with its direct image sheaf on the affine fan $\Delta := \langle\sigma\rangle$ (with respect to the refinement mapping $\Sigma \to \langle\sigma\rangle$). Then the decomposition $F^\bullet_\sigma = E^\bullet_\sigma \oplus K^\bullet$ corresponds to a decomposition $\mathcal{F}^\bullet \cong \mathcal{E}^\bullet \oplus \mathcal{K}^\bullet$ with $\mathcal{E}^\bullet \cong {}_o\mathcal{E}^\bullet$ and the "skyscraper" sheaf $\mathcal{K}^\bullet := {}_\sigma\mathcal{E}^\bullet \otimes K^\bullet$ supported by $\{\sigma\}$. In particular, there is an inclusion

$$E^\bullet_{(\sigma,\partial\sigma)} \subset F^\bullet_{(\sigma,\partial\sigma)} = E^\bullet_{(\sigma,\partial\sigma)} \oplus K^\bullet \ ,$$

and $F^\bullet_{(\sigma,\partial\sigma)}$ is a free $A^\bullet$-module.

We thus obtain a natural commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^\bullet_{(\sigma,\partial\sigma)} & \longrightarrow & E^\bullet_\sigma & \longrightarrow & E^\bullet_{\partial\sigma} & \longrightarrow & 0 \\ & & \cap & & \cap & & \| & & \\ 0 & \longrightarrow & F^\bullet_{(\sigma,\partial\sigma)} & \xrightarrow{\lambda} & F^\bullet_\sigma & \longrightarrow & F^\bullet_{\partial\sigma} & \longrightarrow & 0 \end{array}$$

consisting of free resolutions of the $A^\bullet$-module $E^\bullet_{\partial\sigma} \cong F^\bullet_{\partial\sigma}$.

Using the very definition of $\mathrm{Tor}^{A^\bullet}(*, \mathbf{R}^\bullet)$ and the fact that $\overline{E}^\bullet_{(\sigma,\partial\sigma)} \to \overline{E}^\bullet_\sigma$ is the zero map since $\overline{E}^\bullet_\sigma \to \overline{E}^\bullet_{\partial\sigma}$ is an isomorphism, we obtain identifications

$$(5.3.2) \qquad \overline{E}^\bullet_{(\sigma,\partial\sigma)} \cong \mathrm{Tor}_1(E^\bullet_{\partial\sigma}, \mathbf{R}^\bullet) \cong \ker(\overline{\lambda}) \quad \text{and} \quad \overline{E}^\bullet_\sigma \cong \mathrm{coker}(\overline{\lambda}) \cong \overline{E}^\bullet_{\partial\sigma} \ .$$

On the other hand, we may rewrite $F^\bullet_{(\sigma,\partial\sigma)} = gF^\bullet_\sigma \cong F^\bullet_\sigma[-2]$, where $g \in \mathcal{A}^2(\Sigma)$ is some piecewise linear function on Σ with $\partial\sigma$ as zero set: Write $A^\bullet = B^\bullet[T]$ with $B^\bullet := S^\bullet((V/L)^*) \subset A^\bullet$, such that the kernel of $T \in A^2$ meets σ only in 0. Then, for $\tau \in \partial\sigma$, we set $g_{\hat\tau} = T - f_\tau$, where $f_\tau \in A^2_{\hat\tau} = A^2$ coincides with T on τ and is constant on parallels to L, i.e., $f_\tau \in B^\bullet$.

Now note that

$$E^\bullet_{(\sigma,\partial\sigma)} \subset E^\bullet_{(\sigma,\partial\sigma)} \oplus K^\bullet = F^\bullet_{(\sigma,\partial\sigma)} = gF^\bullet_\sigma \cong F^\bullet_\sigma[-2] \xrightarrow{\eta[-2]} A^\bullet[-2n]$$

defines the homomorphism $\vartheta\colon \overline{E}^\bullet_{(\sigma,\partial\sigma)} \to \overline{F}^\bullet_\sigma[-2]$ mentioned above and an evaluation map for $E^\bullet_{(\sigma,\partial\sigma)}$, and that $K^\bullet \subset F^\bullet_{(\sigma,\partial\sigma)}$ is contained in the kernel of the map $F^\bullet_{(\sigma,\partial\sigma)} \to A^\bullet[-2n]$, since $\overline{K}^q = 0$ for $q \geq 2n$ because of the isomorphism $\overline{E}^{2n}_{(\sigma,\partial\sigma)} \cong \mathbf{R} \cong \overline{F}^{2n}_{(\sigma,\partial\sigma)} \cong \overline{F}^{2n-2}_\sigma$ and the vanishing $\overline{F}^q_{(\sigma,\partial\sigma)} = 0$ for $q > 2n$. Next we remark that, although the first part of the diagram

$$\begin{array}{ccccc} E^\bullet_\sigma \times E^\bullet_{(\sigma,\partial\sigma)} & \longrightarrow & E^\bullet_{(\sigma,\partial\sigma)} & \xrightarrow{\varepsilon} & A^\bullet[-2n] \\ \cap & & \cap & & \| \\ F^\bullet_\sigma \times F^\bullet_{(\sigma,\partial\sigma)} & \longrightarrow & F^\bullet_{(\sigma,\partial\sigma)} & \xrightarrow{\eta[-2]} & A^\bullet[-2n] \end{array}$$

need not be commutative ($E^\bullet_\sigma$ is not necessarily closed under the intersection product in $F^\bullet_\sigma$), commutativity holds after evaluation (where the two evaluation maps are scaled in such a way that the right square is commutative). This is true since the difference of the products in the first and second row is an element in $K^\bullet$, according to the construction.

As the intersection product in $F^\bullet_\sigma$ is $\mathcal{A}^\bullet(\Sigma)$-linear, we may replace $F^\bullet_{(\sigma,\partial\sigma)}$ with $F^\bullet_\sigma[-2]$ and arrive at the following pairing of $A^\bullet$-modules

$$E^\bullet_\sigma \times E^\bullet_{(\sigma,\partial\sigma)} \longrightarrow F^\bullet_\sigma \times F^\bullet_\sigma[-2] \longrightarrow F^\bullet_\sigma[-2] \longrightarrow A^\bullet[-2n]\ .$$

Passing to the quotients modulo $\mathbf{m}_A$, we obtain (5.3.1), with $\mu\colon \overline{F}^\bullet_\sigma[-2] \to \overline{F}^\bullet_\sigma$ being induced by multiplication with the function $g \in \mathcal{A}^\bullet(\Sigma)$. □

5.3b Lemma. *If Poincaré duality holds for affine fans $\langle\sigma\rangle$ with a cone σ of dimension $\le n$, then it also holds for every quasi-convex fan Δ in dimension n.*

Proof: To simplify notation, we introduce the abbreviation $\widetilde{A}^\bullet := A^\bullet[-2n]$. We embed the "global" duality homomorphism

$$\Phi\colon E^\bullet_\Delta \longrightarrow \mathrm{Hom}_{A^\bullet}(E^\bullet_{(\Delta,\partial\Delta)}, \widetilde{A}^\bullet)$$

induced by the pairing (5.1.3) into a commutative diagram of the following form:

$$\begin{array}{ccccccc}
0 & \longrightarrow & E^\bullet_\Delta & \longrightarrow & C^0(\Delta,\partial\Delta;\mathcal{E}^\bullet) & \longrightarrow & C^1(\Delta,\partial\Delta;\mathcal{E}^\bullet) \\
(5.3.3) & & \Big\downarrow{\scriptstyle\Phi} & & \Big\downarrow{\scriptstyle\Psi} & & \Big\downarrow{\scriptstyle\Theta} \\
0 & \longrightarrow & \mathrm{Hom}(E^\bullet_{(\Delta,\partial\Delta)}, \widetilde{A}^\bullet) & \xrightarrow{\kappa} & \bigoplus_{\sigma\in\Delta^n} \mathrm{Hom}(E^\bullet_{(\sigma,\partial\sigma)}, \widetilde{A}^\bullet) & \xrightarrow{\lambda} & \bigoplus_{\tau\in\overset{\circ}{\Delta}{}^{n-1}} \mathrm{Hom}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet_\tau[2])
\end{array}$$

where Hom abbreviates $\mathrm{Hom}_{A^\bullet}$ and Ψ, Θ are the duality homomorphisms corresponding to the dual pairings $E^\bullet_\sigma \times E^\bullet_{(\sigma,\partial\sigma)} \to E^\bullet_{(\sigma,\partial\sigma)} \to \tilde{A}^\bullet$ for $\sigma \in \Delta^n$ and $E^\bullet_\tau \times E^\bullet_{(\tau,\partial\tau)} \to E^\bullet_{(\tau,\partial\tau)} \to \tilde{A}^\bullet_\tau[2]$ with suitably chosen evaluation maps. The upper row is exact, while the lower one is a complex with an injection κ. Since Ψ and Θ turn out to be isomorphisms, a simple diagram chase yields that the same holds for Φ which proves the theorem.

The upper sequence is exact, since Δ is quasi-convex, see Theorem 3.8. Now the evaluation map $\varepsilon\colon E^\bullet_{(\Delta,\partial\Delta)} \to \widetilde{A}^\bullet$ induces a system $(\varepsilon_\sigma)_{\sigma\in\Delta^n}$ of evaluation maps $\varepsilon_\sigma\colon E^\bullet_{(\sigma,\partial\sigma)} \subset E^\bullet_{(\Delta,\partial\Delta)} \to \widetilde{A}^\bullet$:We have to show $\varepsilon_\sigma \neq 0$ for all $\sigma \in \Delta$ resp. that the homomorphism $E^\bullet_{(\sigma,\partial\sigma)} \to E^\bullet_{(\Delta,\partial\Delta)}$ induces a non-trivial map $\mathbf{R} \cong \overline{E}^{2n}_{(\sigma,\partial\sigma)} \to \overline{E}^{2n}_{(\Delta,\partial\Delta)} \cong \mathbf{R}$. In order to do that we embed Δ into a complete fan $\tilde{\Delta}$ and show the corresponding fact for $\overline{E}^{2n}_{(\sigma,\partial\sigma)} \to \overline{E}^{2n}_{\tilde{\Delta}}$. But if $\overline{E}^{2n}_{(\sigma,\partial\sigma)} \to \overline{E}^{2n}_{\tilde{\Delta}}$ is non-trivial, so is $\overline{E}^{2n}_{(\sigma,\partial\sigma)} \to \overline{E}^{2n}_{(\Delta,\partial\Delta)}$, i.e., we may assume that Δ is complete.

To the quasi-convex fan $\Delta_0 := \Delta \setminus \{\sigma\}$ corresponds an exact sequence

$$0 \;\to\; E^\bullet_{(\Delta,\Delta_0)} \cong E^\bullet_{(\sigma,\partial\sigma)} \;\to\; E^\bullet_{\Delta} \;\to\; E^\bullet_{\Delta_0} \;\to\; 0 \,.$$

In the induced exact sequence $\overline{E}^{2n}_{(\sigma,\partial\sigma)} \to \overline{E}^{2n}_{\Delta} \to \overline{E}^{2n}_{\Delta_0}$, the last term vanishes according to 4.4, since Δ_0 is not complete. That yields our claim.

The system of duality isomorphisms $E^\bullet_\sigma \to \mathrm{Hom}(E^\bullet_{(\sigma,\partial\sigma)}, \widetilde{A}^\bullet)$ induced by the compositions of the intersection pairings with the $\varepsilon_\sigma, \sigma \in \Delta^n$, provides the isomorphism Ψ. — The map κ associates to a homomorphism $\varphi\colon E^\bullet_{(\Delta,\partial\Delta)} \to \widetilde{A}^\bullet$ its restrictions to the submodules $E^\bullet_{(\sigma,\partial\sigma)}$ of $E^\bullet_{(\Delta,\partial\Delta)}$. It is injective, since $\bigoplus_{\sigma\in\Delta^n} E^\bullet_{(\sigma,\partial\sigma)} \cong E^\bullet_{(\Delta,\Delta^{\le n-1})}$ is a submodule of maximal rank in $E^\bullet_{(\Delta,\partial\Delta)}$: For $h := \prod_{\tau\in\overset{\circ}{\Delta}{}^{n-1}} h_\tau$, where $h_\tau \in A^2 \setminus \{0\}$ with kernel V_τ, we have

$$hE^\bullet_{(\Delta,\partial\Delta)} \subset E^\bullet_{(\Delta,\Delta^{\le n-1})} \,.$$

This ends the discussion of the first rectangle in (5.3.3).

The map λ is composed of "restriction homomorphisms"

$$\lambda^\sigma_\tau\colon \mathrm{Hom}(E^\bullet_{(\sigma,\partial\sigma)}, \widetilde{A}^\bullet) \;\to\; \mathrm{Hom}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet_\tau[2]), \qquad \varphi \mapsto \varphi_\tau \,,$$

where $\tau \prec_1 \sigma$ is a facet of $\sigma \in \overset{\circ}{\Delta}$. In order to define them, we fix a euclidean norm on V and thus also on $V^* \cong A^2$. Now let $h_\tau \in A^2$ be the unique linear form of length 1 that vanishes on V_τ and is positive on $\overset{\circ}{\sigma}$. Then we use three exact sequences, where the first one is obvious:

$$0 \;\to\; E^\bullet_{(\sigma,\partial\sigma)} \;\to\; E^\bullet_\sigma \;\to\; E^\bullet_{\partial\sigma} \;\to\; 0 \,. \tag{5.3.4}$$

The second one is multiplication with h_τ and the projection onto its cokernel:

$$0 \;\to\; \widetilde{A}^\bullet \xrightarrow{\mu(h_\tau)} \widetilde{A}^\bullet[2] \;\to\; \widetilde{A}^\bullet_\tau[2] \;\to\; 0 \,. \tag{5.3.5}$$

Eventually the subfan $\partial_\tau\sigma := \partial\sigma \setminus \{\tau\}$ of $\partial\sigma$ gives the exact sequence

$$0 \;\to\; E^\bullet_{(\tau,\partial\tau)} \;\to\; E^\bullet_{\partial\sigma} \;\to\; E^\bullet_{\partial_\tau\sigma} \;\to\; 0 \,. \tag{5.3.6}$$

The associated Hom-sequences provide a diagram

$$\begin{tikzcd}
 & & \mathrm{Ext}(E^\bullet_{\partial_\tau\sigma}, \widetilde{A}^\bullet) \arrow[d] & \\
\mathrm{Hom}(E^\bullet_\sigma, \widetilde{A}^\bullet) \arrow[r] & \mathrm{Hom}(E^\bullet_{(\sigma,\partial\sigma)}, \widetilde{A}^\bullet) \arrow[r, "\alpha"] & \mathrm{Ext}(E^\bullet_{\partial\sigma}, \widetilde{A}^\bullet) \arrow[d, "\beta"] & \\
\mathrm{Hom}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet[2]) \arrow[r] & \mathrm{Hom}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet_\tau[2]) \arrow[r, "\gamma"] & \mathrm{Ext}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet) \arrow[r] & \mathrm{Ext}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet[2])
\end{tikzcd} \tag{5.3.7}$$

with $\mathrm{Ext} = \mathrm{Ext}^1_{A^\bullet}$. We show that γ is an isomorphism; we then may set $\lambda^\sigma_\tau := \gamma^{-1} \circ \beta \circ \alpha$. Actually $\mathrm{Ext}^1_{A^\bullet}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet) \to \mathrm{Ext}^1_{A^\bullet}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet[2])$ is the zero homomorphism,

since it is induced by multiplication with h_τ, which annihilates $E^\bullet_{(\tau,\partial\tau)}$. On the other hand, the $A^\bullet_\tau$-module $E^\bullet_{(\tau,\partial\tau)}$ is a torsion module over $A^\bullet$, so that $\mathrm{Hom}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet[2])$ vanishes.

An explicit description of λ^σ_τ is as follows: Let $\varphi\colon E^\bullet_{(\sigma,\partial\sigma)} \to \widetilde{A}^\bullet$ be a homomorphism. Its "restriction" $\lambda^\sigma_\tau(\varphi)$ is the homomorphism $\varphi_\tau\colon E^\bullet_{(\tau,\partial\tau)} \to \widetilde{A}^\bullet_\tau[2]$ given by this rule: For $g \in E^\bullet_{(\tau,\partial\tau)}$, we choose a section $\hat{g} \in E^\bullet_\sigma$ such that $\hat{g}|_{\partial\sigma}$ is the trivial extension of g to $\partial\sigma$; then we have $\varphi_\tau(g) = \varphi(h_\tau\hat{g})|_\tau$.

Let us consider two different cones $\sigma = \sigma_1, \sigma_2 \in \Delta^n$ with intersection $\tau \in \overset{\circ}{\Delta}{}^{n-1}$. From the description of φ_τ, one easily derives that the compositions

$$\text{(5.3.8)} \qquad \mathrm{Hom}(E^\bullet_{(\Delta,\partial\Delta)}, \widetilde{A}^\bullet) \to \mathrm{Hom}(E^\bullet_{(\sigma_i,\partial\sigma_i)}, \widetilde{A}^\bullet) \to \mathrm{Hom}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet_\tau[2])$$

coincide. — Hence, we may define λ in the usual way of a Čech coboundary homomorphism, starting from the appropriate λ^σ_τ's. Thus the lower row of diagram (5.3.3) is a complex.

For the definition of Θ, we need compatible evaluation homomorphisms

$$\varepsilon_\tau\colon E^\bullet_{(\tau,\partial\tau)} \to \widetilde{A}^\bullet_\tau[2]$$

for $\tau \in \overset{\circ}{\Delta}{}^{n-1}$. We choose them as the restrictions of the given evaluation map $E^\bullet_{(\sigma,\partial\sigma)} \to \widetilde{A}^\bullet$ for a σ that includes τ. As we have seen in (5.3.8), ε_τ does not depend on the particular choice of σ. We still have to verify that ε_τ is not the zero homomorphism, i.e., we have to see that λ^σ_τ is injective in degree 0. In diagram (5.3.7), we have to show that α and β are injective in degree 0. By 4.4, the vector spaces $\overline{E}^q_\sigma$ vanish for $q \geq 2n$; hence $E^\bullet_\sigma$ can be generated by elements of degree $< 2n$, and that yields the vanishing of $\mathrm{Hom}(E^\bullet_\sigma; \widetilde{A}^\bullet)$ in degree 0. According to Lemma 5.4 below, the exact sequence

$$0 \to E^\bullet_{(\sigma,\partial_\tau\sigma)} \to E^\bullet_\sigma \to E^\bullet_{\partial_\tau\sigma} \to 0$$

is a free resolution of $E^\bullet_{\partial_\tau\sigma}$, in particular, the module $\mathrm{Ext}^1(E^\bullet_{\partial_\tau\sigma}, \widetilde{A}^\bullet)$ is a quotient of $\mathrm{Hom}(E^\bullet_{(\sigma,\partial_\tau\sigma)}, \widetilde{A}^\bullet)$, and the latter is trivial in degree 0 by 5.4 below. — For $\tau \in \overset{\circ}{\Delta}{}^{n-1}$, the evaluation homomorphisms ε_τ induce isomorphisms

$$E^\bullet_\tau \cong \mathrm{Hom}(E^\bullet_{(\tau,\partial\tau)}, \widetilde{A}^\bullet_\tau[2]) ,$$

which constitute the isomorphism Θ. Finally the commutativity of the second square in (5.3.3) follows from the above explicit description of the restriction homomorphisms λ^σ_τ and the appropriate choice of the evaluation homomorphisms ε_τ. This ends the discussion of (5.3b) and the proof of the theorem. □

For the notation used in the following result that has been used in the proof of 5.3b, we refer to (0.D.2).

5.4 Proposition. *Let σ be a cone of dimension n and $\Lambda \subset \partial\sigma$ be a fan such that $\pi(\Lambda)$ is a quasi-convex subfan of Λ_σ. Then $E^\bullet_{(\sigma,\Lambda)}$ is a free $A^\bullet$-module, and, if in addition Λ is a proper subfan, $\overline{E}^q_{(\sigma,\Lambda)} = 0$ for $q \geq 2n$.*

Proof. As in 0.D, we choose a line $L \subset V$ intersecting $\mathring{\sigma}$ and set $B^\bullet := \pi^*(S^\bullet((V/L)^*)) \subset A^\bullet$; furthermore, we write $A^\bullet = B^\bullet[T]$ with a linear form $T \in A^2$. The exact sequence of $A^\bullet$-modules

$$0 \to E^\bullet_{(\sigma,\Lambda)} \to E^\bullet_\sigma \to E^\bullet_\Lambda \to 0$$

induces an exact Tor-sequence

$$\mathrm{Tor}_2^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet) \to \mathrm{Tor}_1^{A^\bullet}(E^\bullet_{(\sigma,\Lambda)}, \mathbf{R}^\bullet) \to 0 \to \mathrm{Tor}_1^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet) \to \overline{E}^\bullet_{(\sigma,\Lambda)} \to \overline{E}^\bullet_\sigma$$

since $E^\bullet_\sigma$ is free. If $\mathrm{Tor}_2^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet) = 0$, then also $\mathrm{Tor}_1^{A^\bullet}(E^\bullet_{(\sigma,\Lambda)}, \mathbf{R}^\bullet)$, and $E^\bullet_{(\sigma,\Lambda)}$ is a free $A^\bullet$-module by section 0.B. Since the fan $\langle\sigma\rangle$ is not complete, $\overline{E}^q_\sigma = 0$ for $q \geq 2n$; if the same holds for $\mathrm{Tor}_1^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet)$, then it follows for $\overline{E}^q_{(\sigma,\Lambda)}$ as well. It thus remains to determine $\mathrm{Tor}_i^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet)$. Once again we use the exact sequence

$$0 \to \mathbf{R}^\bullet[T][-2] \xrightarrow{\mu(T)} \mathbf{R}^\bullet[T] \to \mathbf{R}^\bullet \to 0$$

of $A^\bullet$-module homomorphisms of degree 0, where $\mathbf{R}^\bullet[T]$ denotes the $A^\bullet$-module $A^\bullet/(\mathbf{m}_{B^\bullet}A^\bullet) = B^\bullet/\mathbf{m}_{B^\bullet}[T]$, and $\mathbf{m}_{B^\bullet} := B^{>0}$, the maximal homogeneous ideal of $B^\bullet$. Using the identity

$$\mathrm{Tor}_i^{B^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet) \cong \mathrm{Tor}_i^{B^\bullet[T]}(E^\bullet_\Lambda, \mathbf{R}^\bullet[T])$$

we obtain the exact sequence

$$\ldots \to \mathrm{Tor}_1^{B^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet[-2]) \to \mathrm{Tor}_1^{B^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet) \to \mathrm{Tor}_1^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet)$$
$$\to E^\bullet_\Lambda \otimes_{B^\bullet} \mathbf{R}^\bullet[-2] \to E^\bullet_\Lambda \otimes_{B^\bullet} \mathbf{R}^\bullet\ ,$$

where $\mathrm{Tor}_i^{B^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet)$ vanishes for $i \geq 1$ since $E^\bullet_\Lambda$ is a free $B^\bullet$-module. This yields the desired description:

$$\mathrm{Tor}_i^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet) = \begin{cases} \ker\Big(\mu(T)\colon E^\bullet_\Lambda \otimes_{B^\bullet} \mathbf{R}^\bullet[-2] \to E^\bullet_\Lambda \otimes_{B^\bullet} \mathbf{R}^\bullet\Big), & \text{if } i = 1; \\ 0, & \text{if } i \geq 2\ . \end{cases}$$

Eventually, if $\pi(\Lambda) \subset \Lambda_\sigma$ is not complete, then the vector space $E^\bullet_\Lambda \otimes_{B^\bullet} \mathbf{R}^\bullet[-2]$ vanishes in degrees $\geq 2n$; hence, the same holds for $\mathrm{Tor}_1^{A^\bullet}(E^\bullet_\Lambda, \mathbf{R}^\bullet)$. □

5.5 Remark. For every purely n-dimensional fan Δ we can define an evaluation map $E^\bullet_{(\Delta,\partial\Delta)} \to A^\bullet[-2n]$ as the restriction of an evaluation map $E^\bullet_{\tilde{\Delta}} \to A^\bullet[-2n]$ for a "completion" $\tilde{\Delta}$ of Δ. It provides a homomorphism $E^\bullet_\Delta \to \mathrm{Hom}(E^\bullet_{(\Delta,\partial\Delta)}, A^\bullet[-2n])$ via the intersection pairing. In accordance with the proof of 5.3b, that is an isomorphism whenever $\tilde{H}^0(\Delta, \partial\Delta; \mathcal{E}^\bullet) = 0$, or equivalently (see 3.7), if $\tilde{H}^0(\Delta_\sigma, \partial\Delta_\sigma; \mathbf{R}^\bullet) =$

$\{0\}$ for each cone $\sigma \in \Delta$. In more geometrical terms, Δ has to be both facet-connected and locally facet-connected, where we call a fan *locally facet-connected* if for each non-zero cone $\sigma \in \Delta$, the fan Δ_σ is facet-connected.

The smallest example of a three-dimensional fan that is both facet-connected and locally facet-connected, but not quasi-convex, is provided by the fan swept out by the "vertical" facets of a triangular prism.

Since the dual pairing $E^\bullet_\Delta \times E^\bullet_{(\Delta,\partial\Delta)} \to A^\bullet[-2n]$ of $A^\bullet$-modules induces a dual pairing of real vector spaces $\overline{E}^\bullet_\Delta \times \overline{E}^\bullet_{(\Delta,\partial\Delta)} \to \mathbf{R}^\bullet[-2n]$, we obtain the following consequence.

5.6 Corollary. *If Δ is a quasi-convex fan of dimension n, then we have*

$$\dim \overline{E}^q_\Delta \;=\; \dim \overline{E}^{2n-q}_{(\Delta,\partial\Delta)}\,. \qquad \square$$

We finally are prepared to prove the "Combinatorial Hard Lefschetz" Theorem 4.6.

Proof of Theorem 4.6: Since f is strictly convex, its graph Γ_f in $V \times \mathbf{R}$ is the support of the boundary fan $\partial\gamma$ of the $(n+1)$-dimensional cone $\gamma := \gamma^+(f)$ in $V \times \mathbf{R}$ as we have seen in 0.D. Denote with $\mathcal{H}^\bullet$ a minimal extension sheaf on $\partial\gamma$. Then $\tau \mapsto \mathcal{H}^\bullet(f(\tau))$ is a minimal extension sheaf on Δ, that we identify with $\mathcal{E}^\bullet$. Then in analogy to (4.3.2) the residue class module of the $A^\bullet[T]$-module $H^\bullet_{\partial\gamma}$ satisfies

$$\overline{H}^\bullet_{\partial\gamma} \cong \overline{E}^\bullet_\Delta / f\overline{E}^\bullet_\Delta = \operatorname{coker}(\overline{\mu} : \overline{E}^\bullet_\Delta[-2] \longrightarrow \overline{E}^\bullet_\Delta)$$

where $\overline{E}^\bullet_\Delta = (A^\bullet/\mathbf{m}) \otimes_{A^\bullet} E^\bullet_\Delta$. Now the vanishing condition $V(\gamma)$ yields the surjectivity of $\overline{\mu}^{2q}$ for $2q \geq n-1$. On the other hand the map μ is selfadjoint with respect to the dual pairing $E^\bullet_\Delta \times E^\bullet_\Delta \to A^\bullet[-2n]$ as well as $\overline{\mu}$ with respect to $\overline{E}^\bullet_\Delta \times \overline{E}^\bullet_\Delta \to \mathbf{R}^\bullet[-2n]$. Hence by Poincaré duality the surjectivity of $\overline{\mu}^{2q}$ for $2q \geq n-1$ implies the injectivity of $\overline{\mu}^{2q}$ for $2q \leq n-1$. $\square$

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Addresses of authors

G. Barthel, L. Kaup
Fachbereich Mathematik
und Statistik
Universität Konstanz
Fach D 203
D-78457 Konstanz
e-mail:
`Gottfried.Barthel@uni-konstanz.de`
`Ludger.Kaup@uni-konstanz.de`

K.-H. Fieseler
Matematiska Institutionen
Box 480
Uppsala Universitet
SE-75106 Uppsala
e-mail: `khf@math.uu.se`

J.P. Brasselet.
IML/CNRS, Luminy Case 907
F-13288 Marseille Cedex 9
e-mail: `jpb@iml.univ-mrs.fr`